

REPRESENTATION THEORY OF POLYADIC GROUPS

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ABSTRACT. In this article, we introduce the notion of representations of polyadic groups and we investigate the connection between these representations and those of retract groups and covering groups.

1. INTRODUCTION

A non-empty set G together with an n -ary operation $f : G^n \rightarrow G$ is called an n -ary groupoid and is denoted by (G, f) . We will assume that $n > 2$.

According to the general convention used in the theory of n -ary systems, the sequence of elements x_i, x_{i+1}, \dots, x_j is denoted by x_i^j . In the case $j < i$ it is the empty symbol. If $x_{i+1} = x_{i+2} = \dots = x_{i+t} = x$, then instead of x_{i+1}^{i+t} we write $\overset{(t)}{x}$. In this convention $f(x_1, \dots, x_n) = f(x_1^n)$ and

$$f(x_1, \dots, x_i, \underbrace{x, \dots, x}_t, x_{i+t+1}, \dots, x_n) = f(x_1^i, \overset{(t)}{x}, x_{i+t+1}^n).$$

An n -ary groupoid (G, f) is called (i, j) -associative, if

$$(1.1) \quad f(x_1^{i-1}, f(x_i^{n+i-1}), x_{n+i}^{2n-1}) = f(x_1^{j-1}, f(x_j^{n+j-1}), x_{n+j}^{2n-1})$$

holds for all $x_1, \dots, x_{2n-1} \in G$. If this identity holds for all $1 \leq i < j \leq n$, then we say that the operation f is associative and (G, f) is called an n -ary semigroup.

If, for all $x_0, x_1, \dots, x_n \in G$ and fixed $i \in \{1, \dots, n\}$, there exists an element $z \in G$ such that

$$(1.2) \quad f(x_1^{i-1}, z, x_{i+1}^n) = x_0,$$

then we say that this equation is i -solvable or solvable at the place i . If this solution is unique, then we say that (1.2) is uniquely i -solvable.

An n -ary groupoid (G, f) uniquely solvable for all $i = 1, \dots, n$, is called an n -ary quasigroup. An associative n -ary quasigroup is called an n -ary group or a polyadic group. In the binary case (i.e., for $n = 2$) it is a usual group.

Now, such and similar n -ary systems have many applications in different branches. For example, in the theory of automata, (cf. [11]), n -ary

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semigroups and n -ary groups are used, some n -ary groupoids are applied in the theory of quantum groups (cf. [15]). Different applications of ternary structures in physics are described by R. Kerner (cf. [13]). In physics there are used also such structures as n -ary Filippov algebras (cf. [16]) and n -Lie algebras (cf. [18]).

The idea of investigations of such groups seems to be going back to E. Kasner's lecture [12] at the fifty-third annual meeting of the American Association for the Advancement of Science in 1904. But the first paper concerning the theory of n -ary groups was written (under inspiration of Emmy Noether) by W. Dörnte in 1928 (see [2]). In this paper Dörnte observed that any n -ary groupoid (G, f) of the form $f(x_1^n) = x_1 \circ x_2 \circ \dots \circ x_n \circ b$, where (G, \circ) is a group and b is its fixed element belonging to the center of (G, \circ) , is an n -ary group. Such n -ary groups, called b -derived from the group (G, \circ) , are denoted by $der_b(G, \circ)$. In the case when b is the identity of (G, \circ) we say that such n -ary group is *reducible* to the group (G, \circ) or *derived* from (G, \circ) . But for every $n > 2$ there are n -ary groups which are not derived from any group. An n -ary group (G, f) is derived from some group iff it contains an element e (called an *n -ary identity*) such that

$$(1.3) \quad f(\overset{(i-1)}{e}, x, \overset{(n-i)}{e}) = x$$

holds for all $x \in G$ and $i = 1, \dots, n$.

It is worthwhile to note that in the definition of an n -ary group, under the assumption of the associativity of the operation f , it suffices only to postulate the existence of a solution of (1.2) at the places $i = 1$ and $i = n$ or at one place i other than 1 and n (cf. [17], p. 213¹⁷). Other useful characterizations of n -ary groups one can find in [3] and [6].

From the definition of an n -ary group (G, f) , we can directly see that for every $x \in G$, there exists only one $z \in G$ satisfying the equation

$$(1.4) \quad f(\overset{(n-1)}{x}, z) = x.$$

This element is called *skew* to x and is denoted by \bar{x} . In a ternary group ($n = 3$) derived from the binary group (G, \cdot) the skew element coincides with the inverse element in (G, \circ) . Thus, in some sense, the skew element is a generalization of the inverse element in binary groups. Dörnte proved (see [2]) that in ternary groups we have $f(x, y, z) = f(\bar{z}, \bar{y}, \bar{x})$ and $\bar{\bar{x}} = x$, but for $n > 3$ this is not true. For $n > 3$ there are n -ary groups in which one fixed element is skew to all elements (cf. [4]) and n -ary groups in which any element is skew to itself.

Nevertheless, the concept of skew elements plays a crucial role in the theory of n -ary groups. Namely, as Dörnte proved (see also [6]), the following theorem is true.

Theorem 1.1. *In any n -ary group (G, f) the following identities*

$$(1.5) \quad f(\overset{(i-2)}{x}, \bar{x}, \overset{(n-i)}{x}, y) = f(y, \overset{(n-j)}{x}, \bar{x}, \overset{(j-2)}{x}) = y,$$

$$(1.6) \quad f\left(\begin{smallmatrix} (k-1) \\ x \end{smallmatrix}, \bar{x}, \begin{smallmatrix} (n-k) \\ x \end{smallmatrix}\right) = x$$

hold for all $x, y \in G$, $2 \leq i, j \leq n$ and $1 \leq k \leq n$.

One can prove (cf. [3]) that for $n > 2$ an n -ary group can be defined as an algebra $(G, f, \bar{})$ with one associative n -ary operation f and one unary operation $\bar{}: x \rightarrow \bar{x}$ satisfying for some $2 \leq i, j \leq n$ the identities (1.5). This means that a non-empty subset H of an n -ary group (G, f) is its subgroup iff it is closed with respect to the operation f and $\bar{x} \in H$ for every $x \in H$.

Fixing in an n -ary operation f all inner elements a_2, \dots, a_{n-1} we obtain a new binary operation

$$x * y = f(x, a_2^{n-1}, y).$$

Such obtained groupoid $(G, *)$ is called a *retract* of (G, f) . Choosing different elements a_1, \dots, a_{n-1} we obtain different retracts. Retracts of n -ary groups are groups. Retracts of a fixed n -ary group are isomorphic (cf. [8]). So, we can consider only retracts of the form

$$x * y = f(x, \begin{smallmatrix} (n-2) \\ a \end{smallmatrix}, y).$$

Such retracts will be denoted by $Ret_a(G, f)$, or simply by $Ret_a(G)$. The identity of the group $Ret_a(G)$ is \bar{a} . One can verify that the inverse element to x has the form

$$(1.7) \quad x^{-1} = f(\bar{a}, \begin{smallmatrix} (n-3) \\ x \end{smallmatrix}, \bar{x}, \bar{a}).$$

Binary retracts of an n -ary group (G, f) are commutative only in the case when there exists an element $a \in G$ such that

$$f(x, \begin{smallmatrix} (n-2) \\ a \end{smallmatrix}, y) = f(y, \begin{smallmatrix} (n-2) \\ a \end{smallmatrix}, x)$$

holds for all $x, y \in G$. An n -ary group with this property is called *semiabelian*. It satisfies the identity

$$(1.8) \quad f(x_1^n) = f(x_n, x_2^{n-1}, x_1)$$

(cf. [3]).

One can prove (cf. [9]) that a semiabelian n -ary group is *medial*, i.e., it satisfies the identity

$$(1.9) \quad f(f(x_{11}^{1n}), f(x_{21}^{2n}), \dots, f(x_{n1}^{nn})) = f(f(x_{11}^{n1}), f(x_{12}^{n2}), \dots, f(x_{1n}^{nn})).$$

In such n -ary groups

$$(1.10) \quad \overline{f(x_1, x_2, x_3, \dots, x_n)} = f(\bar{x}_1, \bar{x}_2, \bar{x}_3, \dots, \bar{x}_n)$$

for all $x_1, \dots, x_n \in G$.

Any n -ary group can be uniquely described by its retract and some automorphism of this retract. Namely, the following Hosszú-Gluskin Theorem (cf. [5] or [7]) is valid.

Theorem 1.2. *An n -ary groupoid (G, f) is an n -ary group iff*

- (1) *on G one can define an operation \cdot such that (G, \cdot) is a group,*

- (2) *there exist an automorphism φ of (G, \cdot) and $b \in G$ such that $\varphi(b) = b$,*
- (3) *$\varphi^{n-1}(x) = b \cdot x \cdot b^{-1}$ for every $x \in G$,*
- (4) *$f(x_1^n) = x_1 \cdot \varphi(x_2) \cdot \varphi^2(x_3) \cdot \dots \cdot \varphi^{n-1}(x_n) \cdot b$ for all $x_1, \dots, x_n \in G$.*

One can prove that $(G, \cdot) = \text{Ret}_a(G, f)$ for some $a \in G$. In connection with this we say that an n -ary group (G, f) is (φ, b) -derived from the group (G, \cdot) .

The main aim of this article is to introduce *representations* of n -ary groups and to investigate their main properties, with a special focus on ternary groups. Note that, this is not the first attempt to study representations of n -ary groups, because there are some other articles, with different point of views concerning representations on n -ary groups, (cf. [1], [10], [17] and [19]). However, our method seems to be the most natural generalization of the notion of representation from binary to n -ary groups.

2. ACTION OF AN n -ARY GROUP ON A SET

Suppose that (G, f) is an n -ary group and A is a non-empty set. We say that (G, f) *acts* on A if for all $x \in G$ and $a \in A$ corresponds a unique element $x.a \in A$ such that

- (i) $f(x_1^n).a = x_1.(x_2.(x_3. \dots .(x_n.a)) \dots)$ for all $x_1, \dots, x_n \in G$,
- (ii) for all $a \in A$, there exists $x \in G$ such that $x.a = a$,
- (iii) the map $a \mapsto x.a$ is a bijection for all $x \in G$.

For $a \in A$, we define the *stabilizer* G_a of a as follows

$$G_a = \{x \in G : x.a = a\}.$$

Proposition 2.1. *G_a is an n -ary subgroup of (G, f) .*

Proof. By condition (ii) of the above definition G_a is non-empty. Since for $x_1, x_2, \dots, x_n \in G_a$ we have

$$f(x_1^n).a = x_1.(x_2.(x_3. \dots .(x_n.a)) \dots) = a,$$

$f(x_1^n) \in G_a$. Hence G_a is closed with respect to the operation f .

Now if $x \in G_a$, then by (1.6) we obtain

$$a = x.a = f(\bar{x}, \overset{(n-1)}{x}).a = \bar{x}.(x. \dots .x.(x.a)) \dots = \bar{x}.a,$$

which implies $\bar{x} \in G_a$. This completes the proof. \square

Proposition 2.2. *If an n -ary group (G, f) acts on a set A , then the relation \sim defined on A by*

$$a \sim b \iff \exists x \in G : x.a = b$$

is an equivalence relation.

Proof. For each $a \in A$ there is $x \in G$ such that $x.a = a$, so $a \sim a$. If $a \sim b$ for $a, b \in A$, then $z.a = b$ for some $z \in G$. Let y be the unique solution of the equation

$$f(y, z, \overset{(n-2)}{x}) = x,$$

where $x \in G$ is such that $x.a = a$. For this y we have $y.b = a$ since

$$a = x.a = f(y, z, \overset{(n-2)}{x}).a = y.z.a = y.b.$$

Thus $b \sim a$. Finally, let $a \sim b$ and $b \sim c$. Then there are $x, y, z \in G$ such that $x.a = b$, $y.b = c$ and $z.b = b$. In this case for $u = f(y, \overset{(n-2)}{z}, x)$ we have

$$u.a = f(y, \overset{(n-2)}{z}, x).a = y.b = c,$$

which proves $a \sim c$. \square

Theorem 2.3. *The formula $x.a = f(x, a, \overset{(n-3)}{x}, \bar{x})$ defines an action of an n -ary group G on itself.*

Proof. The last condition of Theorem 1.2 can be written in the form

$$f(x_1^n) = x_1 \cdot \varphi(x_2) \cdot \varphi^2(x_3) \cdot \dots \cdot \varphi^{n-2}(x_{n-1}) \cdot b \cdot x_n.$$

Thus $\bar{x} = (\varphi(x) \cdot \varphi^2(x) \cdot \dots \cdot \varphi^{n-2}(x) \cdot b)^{-1}$. Consequently

$$(2.1) \quad x.a = x \cdot \varphi(a) \cdot \varphi(x^{-1}).$$

Hence

$$\begin{aligned} y.(x.a) &= y \cdot \varphi(x) \cdot \varphi^2(a) \cdot \varphi^2(x^{-1}) \cdot \varphi(y)^{-1} \\ &= y \cdot \varphi(x) \cdot \varphi^2(a) \cdot \varphi((y \cdot \varphi(x))^{-1}). \end{aligned}$$

Iterating this procedure we obtain

$$\begin{aligned} &x_1.(x_2.(x_3 \dots (x_n.a)) \dots) = \\ &x_1 \cdot \varphi(x_2) \cdot \varphi^2(x_3) \cdot \dots \cdot \varphi^{n-1}(x_n) \cdot \varphi^n(a) \cdot \varphi((x_1 \cdot \varphi(x_2) \cdot \varphi^2(x_3) \cdot \dots \cdot \varphi^{n-1}(x_n))^{-1}). \end{aligned}$$

Since $\varphi^n(a) = b \cdot \varphi(a) \cdot b^{-1}$ from the above we obtain

$$x_1.(x_2.(x_3 \dots (x_n.a)) \dots) = f(x_1^n) \cdot \varphi(a) \cdot \varphi(f(x_1^n)^{-1}).$$

This by (2.1) gives $f(x_1^n).a = x_1.(x_2.(x_3 \dots (x_n.a)) \dots)$. \square

Proposition 2.4. *In semiabelian n -ary groups the relation*

$$a \sim b \iff \exists x \in G : f(x, a, \overset{(n-3)}{x}, \bar{x}) = b$$

is a congruence.

Proof. Indeed, by Proposition 2.2 it is an equivalence relation. To prove that it is a congruence let $a_i \sim b_i$, i.e., $f(x_i, a_i, \overset{(n-3)}{x_i}, \bar{x}_i) = b_i$ for some $x_i \in G$ and all $i = 1, \dots, n$. Then

$$f(b_1^n) = f(f(x_1, a_1, \overset{(n-3)}{x_1}, \bar{x}_1), f(x_2, a_2, \overset{(n-3)}{x_2}, \bar{x}_2), \dots, f(x_n, a_n, \overset{(n-3)}{x_n}, \bar{x}_n)),$$

which by the mediality and (1.10) gives

$$f(b_1^n) = f(f(x_1^n), f(a_1^n), \underbrace{f(x_1^n), \dots, f(x_1^n)}_{n-3}, \overline{f(x_1^n)}).$$

Thus $f(a_1^n) \sim f(b_1^n)$. □

Remark 2.5. The formula (2.1) says that in n -ary groups b -derived from a group (G, \cdot) the above relation coincides with the conjugation in (G, \cdot) . Thus in non-semiabelian n -ary groups it may not be a congruence.

Elements belonging to the same equivalence class are called *conjugate*. The equivalence classes are called *conjugate classes* of an n -ary group G and have the form

$$Cl_G(a) = \{f(x, a, \overset{(n-3)}{x}, \overline{x}) : x \in G\}.$$

As a simple consequence of (1.9) and (1.10) we obtain

Proposition 2.6. *In semiabelian n -ary group the set containing all elements of G conjugated with elements of a given n -ary subgroup also is an n -ary subgroup.*

For $a \in G$, we define the *centralizer* of a , as follows

$$C_G(a) = \{x \in G : f(x, a, \overset{(n-3)}{x}, \overline{x}) = a\}.$$

From Theorem 1.1 it follows that in n -ary groups b -derived from a group (G, \cdot) the centralizer of any $a \in G$ coincides with the centralizer of a in (G, \cdot) .

Proposition 2.7. *For every $x \in C_G(a)$ and every $0 \leq i, j, k \leq n-2$ such that $i + j + k = n-2$ we have*

$$f(\overset{(i)}{x}, a, \overset{(j)}{x}, \overline{x}, \overset{(k)}{x}) = f(\overset{(i)}{x}, \overline{x}, \overset{(j)}{x}, a, \overset{(k)}{x}) = a.$$

Proof. For every $x \in C_G(a)$, we have $f(x, a, \overset{(n-3)}{x}, \overline{x}) = a$. Multiplying this equation on the left by x and on the right by $x, \dots, x, \overline{x}$ ($n-2$ elements), we obtain

$$f(x, f(x, a, \overset{(n-3)}{x}, \overline{x}), \overset{(n-3)}{x}, \overline{x}) = f(x, a, \overset{(n-3)}{x}, \overline{x}) = a,$$

which in view of the associativity of the operation f and (1.6) gives

$$f(x, x, a, \overset{(n-4)}{x}, \overline{x}) = a.$$

Repeating this procedure we obtain

$$f(\overset{(i)}{x}, a, \overset{(n-i-2)}{x}, \overline{x}) = a$$

for every $1 \leq i \leq n-2$. Theorem 1.1 completes the proof. □

3. G-MODULES AND REPRESENTATIONS

All vector spaces in this section are defined over the field of complex numbers and have finite dimension.

Definition 3.1. Suppose that an n -ary group G acts on a vector space V and we have

- (1) $x.(\lambda v + u) = \lambda x.v + x.u,$
- (2) $\exists p \in G \forall v \in V : p.v = v.$

Then we call (V, p) , or simply V , a G -module.

Notions, such as G -submodule, G -homomorphism, irreducibility and so on, are defined by the ordinary way.

Definition 3.2. A map $\Lambda : G \rightarrow GL(V)$ with the property

$$\Lambda(f(x_1^n)) = \Lambda(x_1)\Lambda(x_2) \dots \Lambda(x_n)$$

is a *representation* of G , provided that $\ker \Lambda$ is non-empty. The function

$$\chi(x) = \text{Tr } \Lambda(x)$$

is called the corresponding *character* of Λ .

Remark 3.3. If V is a G -module, then Λ defined by

$$\Lambda(x)(v) = x.v$$

is a representation of G . The converse is also true.

Example 3.4. Let A be an arbitrary binary group with a normal subgroup H . Let $a \in A \setminus H$ be an involution. Then $G = aH$ with the operation

$$f(x, y, z) = xyz$$

is a ternary group. If Λ is an ordinary representation of A with the property $a \in \ker \Lambda$, then, clearly Λ is also a representation of G . For example, suppose $A = GL_n(\mathbb{C})$ and $H = SL_n(\mathbb{C})$. Let $a = \text{diag}(-1, 1, \dots, 1)$ and define $G = aH$. Then, every representation of A in which $a \in \ker \Lambda$ is also a representation of a ternary group G .

Example 3.5. For any subgroup H of an ordinary group A and any element $a \in Z(A) \setminus H$ with the order n we define on $G = aH$ an n -ary operation by

$$f(x_1, x_2, \dots, x_n) = ax_1x_2 \dots x_n.$$

This operation is associative, because $a \in Z(A)$. Also, G is closed under this operation, since $o(a) = n$. So, G is an n -ary group. Any A -representation Λ with $a \in \ker \Lambda$ is also a G -representation.

Example 3.6. The set $G = \mathbb{Z}_n$ with the ternary operation

$$f(x, y, z) = x - y + z \pmod{n}$$

is, by Theorem 1.2, a ternary group. We want to classify all representations of G .

Let $\Lambda : G \rightarrow GL_m(\mathbb{C})$ be any representation. Then we have

$$\Lambda(f(x, y, z)) = \Lambda(x)\Lambda(y)\Lambda(z),$$

equivalently,

$$\Lambda(x - y + z) = \Lambda(x)\Lambda(y)\Lambda(z).$$

We have

$$\Lambda(x + y) = \Lambda(x)\Lambda(0)\Lambda(y), \quad \Lambda(x - y) = \Lambda(x)\Lambda(y)\Lambda(0).$$

Suppose $A = \Lambda(0)$. We have

$$\Lambda(x + y) = \Lambda(x)A\Lambda(y).$$

It is easy to see that $A^2 = I$. Now, define $\Lambda'(x) = A\Lambda(x)$. Then

$$\Lambda'(x + y) = \Lambda'(x)\Lambda'(y),$$

and so, Λ' is an ordinary representation of $(\mathbb{Z}_n, +)$. Hence, every representation of the ternary group G is of the form $\Lambda(x) = A\Lambda'(x)$, where A is an involution and Λ' is an ordinary representation of $(\mathbb{Z}_n, +)$.

Similarly, we can classify all representations of ternary groups of the form $G = (A, f)$, where A is an ordinary abelian group and

$$f(x, y, z) = x - y + z.$$

Theorem 3.7. (Maschke) *Let G be a finite n -ary group. Then every G -module is completely reducible.*

Proof. Let (V, p) be a G -module and $W \leq_G V$. Suppose $V = W \oplus X$, where X is just a subspace. Let $\varphi : V \rightarrow W$ be the corresponding projection. Define a new map $\theta : V \rightarrow V$ as

$$\theta(v) = \frac{1}{|G|} \sum_{x \in G} \bar{x} \cdot \varphi(x.v).$$

It is easy to see that

$$\theta(x.v) = x.p. \dots .p.\theta(v) = x.\theta(v).$$

So θ is a G -homomorphism and hence its kernel is a G -submodule. For all $w \in W$, we have $\theta(w) = w$ and so $\theta^2 = \theta$. Now, we have $V = W \oplus \ker \theta$. \square

Remark 3.8. Any G -module (V, p) is also an ordinary $Ret_p(G)$ -module, because

$$(x * y).v = f(x, \overset{(n-2)}{p}, y).v = x.p. \dots .p.y.v = x.y.v.$$

From now on, we will assume that $e \in G$ is an arbitrary fixed element. For all $p \in G$, we have $Ret_e(G) \cong Ret_p(G)$ and further the isomorphism is given by the following rule

$$h(x) = f(\overset{(n-2)}{e}, x, \bar{p}).$$

By \hat{G} we denote the binary group $Ret_e(G)$. If (V, p) is a G -module, then we can define a \hat{G} -module structure on V by $x \circ v = h(x).v$. So, we have

$$x \circ v = f\left(\begin{smallmatrix} (n-2) \\ e \end{smallmatrix}, x, \bar{p}\right).v = e. \dots .e.x.\bar{p}.v.$$

But, we have $\bar{p}.v = \bar{p}.p. \dots .p.v = f\left(\bar{p}, \begin{smallmatrix} (n-1) \\ p \end{smallmatrix}\right).v = p.v = v$. Hence

$$x \circ v = \underbrace{e. \dots .e}_{n-2}.x.v.$$

Now, every G -module is also a \hat{G} -module, but the converse is not true in general. During this article, we will give some necessary and sufficient conditions for a \hat{G} -module to be also a G -module. The next proposition is the first condition of this type.

Proposition 3.9. *Let V be a \hat{G} -module. Then V is a G -module iff*

$$\forall x_2, \dots, x_{n-1} \in G \ \forall v : f(\bar{e}, x_2^{n-1}, \bar{e}).v = x_2.x_3. \dots .x_{n-1}.v.$$

Proof. We have

$$\begin{aligned} f(x_1^n) &= f(f(x_1, \begin{smallmatrix} (n-2) \\ e \end{smallmatrix}, \bar{e}), x_2^n) \\ &= f(x_1, \begin{smallmatrix} (n-2) \\ e \end{smallmatrix}, f(\bar{e}, x_2^n)) \\ &= x_1 * f(\bar{e}, x_2^n) \\ &= x_1 * f(\bar{e}, x_2^{n-1}, f(\bar{e}, \begin{smallmatrix} (n-2) \\ e \end{smallmatrix}, x_n)) \\ &= x_1 * f(\bar{e}, x_2^{n-1}, \bar{e}) * x_n. \end{aligned}$$

So, the equality

$$f(x_1^n).v = x_1.x_2. \dots .x_{n-1}.x_n.v$$

holds, iff

$$f(\bar{e}, x_2^{n-1}, \bar{e}).v = x_2.x_3. \dots .x_{n-1}.v$$

for all x_2, \dots, x_{n-1} and v . □

Remark 3.10. Suppose that V is a G -module in which the corresponding representation is Λ . We know that V is also a \hat{G} -module. The corresponding representation of this last module is

$$\hat{\Lambda}(x) = \underbrace{\Lambda(e) \dots \Lambda(e)}_{n-2} \Lambda(x).$$

Because in \hat{G} , the identity element is \bar{e} , we have

$$\hat{\Lambda}(\bar{e}) = id.$$

So $\Lambda(e)^{n-2} \Lambda(\bar{e}) = id$ and hence

$$\Lambda(\bar{e}) = \Lambda(e)^{2-n}.$$

In the sequel, the corresponding character of $\hat{\Lambda}$, will be denoted by $\hat{\chi}$.

Proposition 3.11. *Suppose that Λ is a representation of G with the character χ . Then χ is fixed on the conjugate classes of G .*

Proof. Indeed, for every $b \in Cl_G(a)$ we have

$$\Lambda(b) = \Lambda(f(x, a, \binom{n-3}{x}, \bar{x})) = \Lambda(x)\Lambda(a)\Lambda(x)^{n-3}\Lambda(\bar{x}),$$

so

$$\begin{aligned} \chi(b) &= Tr(\Lambda(x)\Lambda(a)\Lambda(x)^{n-3}\Lambda(\bar{x})) \\ &= Tr(\Lambda(x)\Lambda(a)\Lambda(e)^{n-2}\Lambda(\bar{e})\Lambda(x)^{n-3}\Lambda(\bar{x})) \\ &= Tr(\Lambda(a)\Lambda(e)^{n-2}\Lambda(\bar{e})\Lambda(x)^{n-3}\Lambda(\bar{x})\Lambda(x)) \\ &= Tr(\Lambda(a)\Lambda(e)^{n-2}\Lambda(f(\bar{e}, \binom{n-3}{x}, \bar{x}, x))) \\ &= Tr(\Lambda(a)\Lambda(e)^{n-2}\Lambda(\bar{e})) \\ &= Tr(\Lambda(a)) \\ &= \chi(a). \end{aligned}$$

This completes the proof. \square

Proposition 3.12. *Suppose that $\Lambda : (G, f) \rightarrow GL(V)$ is a representation of the finite n -ary group (G, f) with the corresponding character χ . Let*

$$\ker \chi = \{x \in G : \chi(x) = \dim V\}.$$

Then $\ker \chi = \ker \Lambda$.

Proof. Let $\dim V = m$. It is clear that $\ker \Lambda \subseteq \ker \chi$. Moreover, for each $x \in G$ of order k we have

$$\Lambda(x)^{m^k} = \Lambda(x).$$

Hence $\Lambda(x)$ is a root of the polynomial $T^{m^k-1} - 1$. But, this polynomial has distinct roots in \mathbb{C} , so $\Lambda(x)$ can be diagonalized, i.e.,

$$\Lambda(x) \sim \text{diag}(\varepsilon_1, \dots, \varepsilon_m),$$

where all ε_i are roots of unity. Now, we have

$$\chi(x) = \varepsilon_1 + \dots + \varepsilon_m.$$

If $\chi(x) = m$, then $\varepsilon_i = 1$ for all i . Hence $\Lambda(x) = id$ and so $x \in \ker \Lambda$. This completes the proof. \square

In the next proposition, we obtain the explicit form of the character $\hat{\chi}$.

Proposition 3.13. *Let χ be a character of an n -ary group (G, f) . Then for any $p \in \ker \chi$ we have*

$$\hat{\chi}(x) = \chi(f(\binom{n-2}{e}, x, \bar{p})).$$

Proof. We know that χ is a character of $Ret_p(G)$. On the other hand there is an isomorphism

$$h : Ret_e(G) \rightarrow Ret_p(G),$$

where $h(x) = f(\binom{n-2}{e}, x, \bar{p})$. So, the composite map $\chi \circ h$ is a character of $Ret_e(G)$. Let Λ be the corresponding representation of χ . Now, we have

$$\begin{aligned} \chi(h(x)) &= Tr(\Lambda(e)^{n-2} \Lambda(x) \Lambda(\bar{p})) \\ &= Tr(\Lambda(e)^{n-2} \Lambda(x)) \\ &= Tr \hat{\Lambda}(x). \end{aligned}$$

Hence $\hat{\chi}(x) = \chi(f(\binom{n-2}{e}, x, \bar{p}))$. \square

Remark 3.14. Now, for any irreducible character χ of an n -ary group (G, f) , we have an ordinary irreducible character $\hat{\chi}$ of the binary group $\hat{G} = Ret_e(G)$. So, we obtain the following orthogonality relation for the irreducible characters of G :

$$\frac{1}{|G|} \sum_{x \in G} \chi_1(f(\binom{n-2}{e}, x, \bar{p}_1)) \overline{\chi_2(f(\binom{n-2}{e}, x, \bar{p}_2))} = \delta_{\hat{\chi}_1, \hat{\chi}_2},$$

where $p_1 \in \ker \chi_1$ and $p_2 \in \ker \chi_2$ are arbitrary elements.

Proposition 3.15. *If a representation $\Gamma : Ret_e(G, f) \rightarrow GL(V)$ is also a representation of the n -ary group (G, f) , then*

$$\Gamma(\bar{x}) = \Gamma(x)^{2-n}$$

for every $x \in G$.

Proof. Indeed, $f(\binom{n-1}{x}, \bar{x}) = x$ implies $\Gamma(x)^{n-1} \Gamma(\bar{x}) = \Gamma(x)$, which gives $\Gamma(\bar{x}) = \Gamma(x)^{2-n}$. \square

Corollary 3.16. *Let (G, f) be a ternary group. Then a representation $\Gamma : Ret_e(G, f) \rightarrow GL(V)$ is also a representation of (G, f) iff*

$$\Gamma(\bar{x}) = \Gamma(x)^{-1}$$

for every $x \in G$.

Proof. From Proposition 3.9 it follows that $\Gamma : Ret_e(G, f) \rightarrow GL(V)$ is a representation of a ternary group (G, f) iff it satisfies the identity

$$\Gamma(f(\bar{e}, x, \bar{e})) = \Gamma(x).$$

If $\Gamma(\bar{x}) = \Gamma(x)^{-1}$ holds for all $x \in G$, then, in view of (1.7), for all $x \in G$ we have

$$\Gamma(f(\bar{e}, x, \bar{e})) = \Gamma(f(\bar{e}, \bar{\bar{x}}, \bar{e})) = \Gamma(\bar{x}^{-1}) = \Gamma(\bar{x})^{-1} = \Gamma(x).$$

Hence Γ is a representation of (G, f) .

The converse statement is a consequence of Proposition 3.15. \square

Remark 3.17. We can use the above proposition to obtain some deeper results in the case when G has a central element. Note that, according to [8], an n -ary group (G, f) has a central element iff it is b -derived from a binary group (G, \cdot) and $b \in Z(G, \cdot)$. Obviously, in this case $Z(G, f) = Z(G, \cdot)$.

Proposition 3.18. *Let e be a central element of an n -ary group $(G, f) = \text{der}_b(G, \cdot)$. Then a representation $\Gamma : \text{Ret}_e(G) \rightarrow GL(V)$ is a representation of (G, f) iff*

$$\Gamma(x_2x_3 \dots x_n e^{2-n}) = \Gamma(x_2)\Gamma(x_3) \dots \Gamma(x_n)$$

for all $x_2, \dots, x_n \in G$.

Proof. Since $(G, f) = \text{der}_b(G, \cdot)$ the binary operation in $\text{Ret}_e(G, f)$ has the form

$$x * y = f(x, \overset{(n-2)}{e}, y) = xy e^{n-2} b.$$

For a representation Γ of $\text{Ret}_e(G, f)$, we have

$$(3.1) \quad \Gamma(x * y) = \Gamma(x)\Gamma(y).$$

Now, for Γ to be a representation of (G, f) , it is necessary and sufficient that

$$\Gamma(f(x_1^n)) = \Gamma(x_1x_2 \dots x_nb) = \Gamma(x_1)\Gamma(x_2) \dots \Gamma(x_n).$$

If we replace in (3.1), y by $x_2 \dots x_n e^{2-n}$, we obtain

$$\Gamma(x_1x_2 \dots x_nb) = \Gamma(x_1)\Gamma(x_2 \dots x_n e^{2-n}).$$

So Γ is a representation of (G, f) , iff

$$\Gamma(x_2x_3 \dots x_n e^{2-n}) = \Gamma(x_2)\Gamma(x_3) \dots \Gamma(x_n)$$

for all $x_2, \dots, x_n \in G$. □

In an n -ary group $(G, f) = \text{der}_b(G, \cdot)$ we have $\bar{x} = x^{2-n}b^{-1}$. Hence, comparing the above result with Proposition 3.15 we obtain

Corollary 3.19. *Let e be a central element of an n -ary group $(G, f) = \text{der}_b(G, \cdot)$. If a representation $\Gamma : \text{Ret}_e(G) \rightarrow GL(V)$ is a representation of (G, f) , then $\Gamma(x^{2-n}b^{-1}) = \Gamma(x)^{2-n}$ for every $x \in G$.*

In the case of ternary groups, by Corollary 3.16, we obtain stronger result.

Corollary 3.20. *Let $(G, f) = \text{der}_b(G, \cdot)$ be a ternary group. Then a representation $\Gamma : \text{Ret}_e(G, f) \rightarrow GL(V)$ is also a representation of (G, f) , iff $\Gamma((bx)^{-1}) = \Gamma(x)^{-1}$ for every $x \in G$.*

Proposition 3.21. *Let e be a central element of an n -ary group $(G, f) = \text{der}_b(G, \cdot)$. Then a character χ of $\text{Ret}_e(G, f)$ is a character of (G, f) iff for all $x \in G$ we have $\chi(\bar{x}) = \overline{\chi(x)}$.*

Proof. Let $\Gamma : \text{Ret}_e(G, f) \rightarrow GL(V)$ be a representation corresponding to χ . If χ is a character of (G, f) , then Γ is also a representation of (G, f) and so $\Gamma(\bar{x}) = \Gamma(x)^{-1}$. Hence we have $\chi(\bar{x}) = \overline{\chi(x)}$.

Conversely, if $\chi(\bar{x}) = \overline{\chi(x)}$ holds for all $x \in G$, then in particular $\overline{\chi(e)} = \chi(\bar{e})$. Thus $\chi(e) = \chi(\bar{e})$ because $\chi(\bar{e})$ is real. Now, for all $x \in G$, we have $x * \bar{x} = f(x, e, \bar{x}) = f(e, x, \bar{x}) = e$, so $\chi(x * \bar{x}) = \chi(e) = \chi(\bar{e})$. Hence,

$$x * \bar{x} \in \ker \chi = \ker \Gamma.$$

This shows that $\Gamma(x^{-1}) = \Gamma(\bar{x})$ and so Γ is a representation of G . Hence χ is also a character of G . \square

Proposition 3.22. *Let e be a central element of a ternary group $(G, f) = \text{der}_b(G, \cdot)$. If χ is a common character of (G, f) and $\text{Ret}_e(G, f)$, then $\hat{\chi} = \chi$.*

Proof. We have $\chi(\bar{e}) = \overline{\chi(e)}$, so $\chi(e)$ is real, and hence $\chi(e) = \chi(\bar{e})$. So $e \in \ker \chi$. Now, suppose $p = e$. Then

$$\hat{\chi}(x) = \chi(f(e, x, \bar{p})) = \chi(f(e, x, \bar{e})) = \chi(f(x, e, \bar{e})) = \chi(x),$$

which completes the proof. \square

In the remaining part of this section, we try to answer this problem: when $\hat{\chi}_1 = \hat{\chi}_2$? We give an answer to this question for n -ary groups with some central elements.

Proposition 3.23. *For an n -ary group (G, f) with a central element e the following assertions are true:*

- (1) *Let (V, p) be a G -module and $h : V \rightarrow V$ be a \hat{G} -homomorphism. Then h is also a G -homomorphism.*
- (2) *Let (V_1, p_1) and (V_2, p_2) be two G -modules and $h : V_1 \rightarrow V_2$ be a \hat{G} -homomorphism. Then h is a G -homomorphism, iff $h(e.v) = e.h(v)$.*
- (3) *Let (V_1, p_1) and (V_2, p_2) be two G -modules and $h : V_1 \rightarrow V_2$ be a \hat{G} -homomorphism. Then h is a G -homomorphism, iff $p_1.h(v) = h(v)$ for every $v \in V_1$.*
- (4) *Let (V_1, p_1) and (V_2, p_2) be two G -modules and*

$$V_1 \cong_{\hat{G}} V_2.$$

Then $V_1 \cong_G V_2$, iff for all $u \in V_2$, $p_1.u = u$.

Proof. (1). In view of $x \circ y = f(x, \binom{n-2}{e}, y)$, for a G -module (V, p) , we have

$$\begin{aligned}
 h(e.v) &= h(f(\binom{n-1}{e}, \bar{e}).v) \\
 &= h(f(f(\binom{n-1}{e}, \bar{e}), \binom{n-1}{p}).v)) \\
 &= h(f(f(e, \binom{n-2}{p}, \bar{e}), \binom{n-2}{e}, p).v)) \\
 &= h(f(e, \binom{n-2}{p}, \bar{e}) \circ v) \\
 &= f(e, \binom{n-2}{p}, \bar{e}) \circ h(v) \\
 &= f(f(e, \binom{n-2}{p}, \bar{e}), \binom{n-2}{e}, p).h(v) \\
 &= f(e, \binom{n-2}{p}, f(\bar{e}, \binom{n-2}{e}, p)).h(v) \\
 &= f(e, \binom{n-1}{p}).h(v) \\
 &= e.p. \dots .p.h(v) \\
 &= e.h(v).
 \end{aligned}$$

Now for all $x \in G$, we have $h(x \circ v) = x \circ h(v)$, so

$$h(\underbrace{e. \dots .e}_{n-2}.x.v) = e. \dots .e.x.h(v).$$

Hence

$$\underbrace{e. \dots .e}_{n-2}.h(x.v) = e. \dots .e.x.h(v).$$

Since the map $u \mapsto e.u$ is bijection, we have $h(x.v) = x.h(v)$.

(2). The proof of this part is just as the above.

(3). Suppose h is a G -homomorphism. Then $p_1.h(v) = h(p_1.v) = h(v)$ for every $v \in V_1$.

Conversely, assume that for all $v \in V_1$ holds $p_1.h(v) = h(v)$. Then

$$\begin{aligned}
 h(e.v) &= h(f(\overset{(n-1)}{e}, \bar{e}). \underbrace{p_1 \dots p_1}_{n-2}.v) \\
 &= h(\underbrace{e \dots e}_{n-1}. \bar{e}. \underbrace{p_1 \dots p_1}_{n-2}.v) \\
 &= h(f(e, \overset{(n-2)}{p_1}, \bar{e}). \underbrace{e \dots e}_{n-2}.v) \\
 &= h(f(e, \overset{(n-2)}{p_1}, \bar{e}) \circ v) \\
 &= f(e, \overset{(n-2)}{p_1}, \bar{e}) \circ h(v) \\
 &= f(f(e, \overset{(n-2)}{p_1}, \bar{e}). \underbrace{e \dots e}_{n-2}.h(v)) \\
 &= f(f(e, \overset{(n-2)}{p_1}, \bar{e}). \underbrace{e \dots e}_{n-2}.p_1.h(v)) \\
 &= f(f(e, \overset{(n-2)}{p_1}, \bar{e}), \overset{(n-2)}{e}, p_1).h(v) \\
 &= f(e, \overset{(n-2)}{p_1}, f(\bar{e}, \overset{(n-2)}{e}, p_1)).h(v) \\
 &= f(e, \overset{(n-1)}{p_1}).h(v) \\
 &= e.h(v).
 \end{aligned}$$

(4). Let $h : V_1 \rightarrow V_2$ be a G -isomorphism. Then h is also a \hat{G} -homomorphism, and hence $p_1.h = h$. Because h is onto, we obtain $p_1.u = u$, for all $u \in V_2$.

Conversely, suppose $p_1.u = u$, for all $u \in V_2$. Let $h : V_1 \rightarrow V_2$ be a \hat{G} -isomorphism. Then $p_1.h = h$, and so h is a G -isomorphism. \square

Proposition 3.24. *Let (G, f) be an n -ary group with a central element and let $\Lambda_1, \Lambda_2 : G \rightarrow GL(V)$ be two representations of (G, f) , such that $\hat{\Lambda}_1 \sim \hat{\Lambda}_2$. Then $\Lambda_1 \sim \Lambda_2$, iff $\ker \Lambda_1 = \ker \Lambda_2$.*

Proof. Let $p \in \ker \Lambda_1 = \ker \Lambda_2$. We define two G -modules V_1 and V_2 , as follows: V_1 is the vector space V with the action $x.v = \Lambda_1(x)(v)$, V_2 is the vector space V with the action $x.v = \Lambda_2(x)(v)$. Then $\hat{\Lambda}_1 \sim \hat{\Lambda}_2$ implies

$$V_1 \cong_{\hat{G}} V_2,$$

and $p.u = u$, for all $u \in V_2$. So, $V_1 \cong_G V_2$. This proves $\Lambda_1 \sim \Lambda_2$.

Conversely, let $\Lambda_1 \sim \Lambda_2$. Hence, we have $V_1 \cong_G V_2$. By the previous proposition, for $p \in \ker \Lambda_1$ and $u \in V_2$, we have $p.u = u$. Thus $\Lambda_2(p) = id$. Therefore, $\ker \Lambda_1 = \ker \Lambda_2$. \square

Corollary 3.25. *Let χ_1 and χ_2 be two characters of an n -ary group (G, f) with a central element e . If $\hat{\chi}_1 = \hat{\chi}_2$, then $\chi_1 = \chi_2$ iff $\chi_1(e) = \chi_2(e)$.*

Proof. Suppose that Λ_1 and Λ_2 are the corresponding representations. So $\hat{\Lambda}_1 \sim \hat{\Lambda}_2$. By the above proposition, $\chi_1 = \chi_2$, iff $\ker \Lambda_1 = \ker \Lambda_2$. But, we have

$$\ker \Lambda_1 = \{x \in G : \hat{\Lambda}_1(x) = \Lambda_1(e)\},$$

$$\ker \Lambda_2 = \{x \in G : \hat{\Lambda}_2(x) = \Lambda_2(e)\}.$$

Hence $\chi_1 = \chi_2$, iff $\Lambda_1(e) \sim \Lambda_2(e)$, and this is equivalent to $\chi_1(e) = \chi_2(e)$. \square

Remark 3.26. In the last two propositions and Corollary 3.25 the assumption that e is a central element can be replaced by the assumption that that an n -ary group (G, f) is semiabelian.

4. CONNECTION WITH THE REPRESENTATIONS OF THE COVERING GROUP

According to Post's Coset Theorem (cf. [17] or [14]) for any n -ary group (G, f) there exists a binary group (G^*, \cdot) and its normal subgroup H such that $G^*/H \simeq \mathbb{Z}_{n-1}$ and $G \subseteq G^*$ and

$$f(x_1^n) = x_1 \cdot x_2 \cdot x_3 \cdot \dots \cdot x_n$$

for all $x_1, \dots, x_n \in G$.

The group (G^*, \cdot) is called the *covering group* for (G, f) . We know several methods of a construction of such group. The smallest covering group has the form $G_a^* = G \times \mathbb{Z}_{n-1}$, where

$$\langle x, r \rangle \cdot \langle y, s \rangle = \langle f_*(x, \overset{(r)}{a}, y, \overset{(s)}{a}, \bar{a}, \overset{(n-2-r \diamond s)}{a}), r \diamond s \rangle,$$

$r \diamond s = (r + s + 1) \pmod{(n-1)}$ and $a \in G$ an arbitrary but fixed element. The symbol f_* means that the operation f is used one or two times (depending on the value s and t). Clearly fixing various element a of G , we obtain various groups but all these groups are isomorphic (cf. [14]).

The element $(\bar{a}, n-2)$ is the identity of the group (G_a^*, \cdot) . The inverse element has the form

$$\langle x, t \rangle^{-1} = \langle f_*(\bar{a}, \overset{(n-2-t)}{a}, \bar{x}, \overset{(n-3)}{x}, \bar{a}, \overset{(t+1)}{a}), k \rangle,$$

where $k = (n-3-t) \pmod{(n-1)}$.

The set G is identified with the subset $\{\langle x, 0 \rangle : x \in G\}$. Every retract of (G, f) is isomorphic to the normal subgroup

$$H = \{\langle x, n-2 \rangle : x \in G\}.$$

Suppose that V is a G_a^* -module. Then for $x_1, \dots, x_n \in G$ we have

$$\begin{aligned}
 x_1.x_2.x_3. \dots .x_n.v &= \langle x_1, 0 \rangle . \langle x_2, 0 \rangle . \langle x_3, 0 \rangle . \dots . \langle x_n, 0 \rangle . v \\
 &= \langle f(x_1, x_2, \bar{a}, \overset{(n-3)}{a}), 1 \rangle . \langle x_3, 0 \rangle . \dots . \langle x_n, 0 \rangle . v \\
 &= \langle f(f(x_1^2, \bar{a}, \overset{(n-3)}{a}), a, x_3, \bar{a}, \overset{(n-4)}{a}), 2 \rangle . \dots . \langle x_n, 0 \rangle . v \\
 &= \langle f(x_1^2, f(\bar{a}, \overset{(n-2)}{a}, x_3), \bar{a}, \overset{(n-4)}{a}), 2 \rangle . \dots . \langle x_n, 0 \rangle . v \\
 &= \langle f(x_1^3, \bar{a}, \overset{(n-4)}{a}), 2 \rangle . \dots . \langle x_n, 0 \rangle . v \\
 &\vdots \\
 &= \langle f(x_1^n), 0 \rangle . v \\
 &= f(x_1^n).v
 \end{aligned}$$

So, we obtain

Proposition 4.1. *Let (G_a^*, \cdot) be the covering group for an n -ary group (G, f) . Then for a G_a^* -module V to be a G -module it is necessary and sufficient that*

$$\exists p \in G \forall v \in V : p.v = v.$$

Hence, we proved

Proposition 4.2. *Let (G_a^*, \cdot) be the covering group for an n -ary group (G, f) . A representation Γ of G_a^* is a representation of G , iff $\ker \Gamma \cap G \neq \emptyset$. If Γ is irreducible G^* -representation, then it is also irreducible as a representation of G .*

Now, suppose (V, p) is a G -module. For the covering group (G_p^*, \cdot) of (G, f) we can define an action of G_p^* on V as

$$(x, k).v = x.v.$$

Then, it can be easily verified that V is a G_p^* -module. But, we know that $G_a^* \cong G_p^*$, so let $h : G_a^* \rightarrow G_p^*$ be any isomorphism. For any $x \in G_a^*$, define $x.v = h(x).v$. Hence V becomes a G_a^* -module. Further, if W is a G -submodule of G , then it is also a G_p^* -submodule and so a G_a^* -submodule. Hence, we proved

Theorem 4.3. *There is a bijection between the set of all irreducible representations of (G, f) and the set of all irreducible representations of G_a^* with kernels not disjoint from G .*

5. NORMAL SUBGROUPS IN POLYADIC GROUPS

In this section, we show that the representation theory of n -ary groups reduces to the representation theory of binary groups. For this we introduce the concept of normal n -ary subgroup.

Definition 5.1. An n -ary subgroup H of an n -ary group (G, f) is called *normal* if

$$f(\overset{(n-3)}{a}, \bar{a}, h, a) \in H$$

for all $h \in H$ and $a \in G$. A normal subgroup $H \neq G$ containing at least two elements is called *proper*. If G has no any proper normal subgroup, then we say that it is *simple*. If $H = G$ is the only simple subgroup of G , then we say it is *strongly simple*.

Definition 5.2. For any n -ary subgroup H of an n -ary group (G, f) we define the relation \sim_H on G , by

$$a \sim_H b \iff \exists x, y \in H : b = f(a, \overset{(n-2)}{x}, y).$$

Such defined relation is an equivalence on G .

Lemma 5.3. $a \sim_H b \iff \exists x_2, \dots, x_n \in H : b = f(a, x_2^n).$

Proof. Indeed, if $b = f(a, x_2^n)$ for some $x_2, \dots, x_n \in H$, then, in view of Theorem 1.1, for every $x \in H$ we have

$$b = f(a, x_2^n) = f(a, f(\overset{(n-2)}{x}, \bar{x}, x_2), x_3^n) = f(a, \overset{(n-2)}{x}, y),$$

where $y = f(\bar{x}, x_2^n) \in H$, so $a \sim_H b$. The converse is obvious. \square

The equivalence class of G , containing a is denoted by aH and is called the *left coset* of H with the representative a . By Lemma 5.3 it has the form

$$aH = \{f(a, \overset{(n-2)}{x}, y) : x, y \in H\} = \{f(a, h_2^n) : h_2, \dots, h_n \in H\}.$$

The n -ary group (G, f) is partitioned by cosets of H .

Proposition 5.4. If H is a finite n -ary subgroup of (G, f) , then for all $a \in G$, we have $|aH| = |H|$.

Proof. By Theorem 1.2, for an n -ary group (G, f) there is a binary group (G, \cdot) , $\varphi \in \text{Aut}(G, \cdot)$ and an element $b \in G$ such that

$$f(x_1^n) = x_1 \cdot \varphi(x_2) \cdot \varphi^2(x_3) \dots \varphi^{n-1}(x_n) \cdot b,$$

for all $x_1, \dots, x_n \in G$. So, we have

$$aH = \{a \cdot \varphi(x_2) \cdot \varphi^2(x_3) \dots \varphi^{n-1}(x_n) \cdot b : x_2, \dots, x_n \in H\}.$$

But, clearly this set is in one-one correspondence with the set

$$\{\varphi(x_2) \cdot \varphi^2(x_3) \dots \varphi^{n-1}(x_n) \cdot b : x_2, \dots, x_n \in H\},$$

which does not depend on a . So, we have $|aH| = |H|$. \square

On the set $G/H = \{aH : a \in G\}$ we introduce the operation

$$f_H(a_1H, a_2H, \dots, a_nH) = f(a_1^n)H.$$

Proposition 5.5. If H is a normal n -ary subgroup of (G, f) , then $(G/H, f_H)$ is an n -ary group derived from the group $\text{Ret}_H(G/H, f)$.

Proof. First we show that the operation f_H is well-defined. For this let $a_i H = b_i H$ for some $a_i, b_i \in G$, $i = 1, 2, \dots, n$. Then

$$b_1 = f(a_1, x_2^n), \quad b_2 = f(a_2, y_2^n), \quad \dots, \quad b_n = f(a_n, z_2^n)$$

for some $x_i, y_i, \dots, z_i \in H$

Now, using Theorem 1.2 we obtain

$$\begin{aligned} f(b_1^n) &= f(f(a_1, x_2^n), f(a_2, y_2^n), \dots, f(a_n, z_2^n)) \\ &= f(f(a_1, x_2^{n-1}, f(\overset{(n-2)}{a_2}, \bar{a}_2, x_n)), f(a_2, y_2^n), \dots, f(a_n, z_2^n)) \\ &= f(f(a_1, x_2^{n-1}, a_2), f(f(\overset{(n-3)}{a_2}, \bar{a}_2, x_n, a_2), y_2^n), \dots, f(a_n, z_2^n)) \\ &= f(f(a_1, x_1^{n-1}, a_2), f(w_n, y_2^n), \dots, f(a_n, z_2^n)) \\ &= f(f(a_1, x_1^{n-2}, f(\overset{(n-2)}{a_2}, \bar{a}_2, x_{n-1}), a_2), f(w_n, y_2^n), \dots, f(a_n, z_2^n)) \\ &= f(f(a_1, x_1^{n-2}, a_2, f(\overset{(n-3)}{a_2}, \bar{a}_2, x_{n-1}, a_2)), f(w_n, y_2^n), \dots, f(a_n, z_2^n)) \\ &= f(f(a_1, x_1^{n-2}, a_2, w_{n-1}), f(w_n, y_2^n), \dots, f(a_n, z_2^n)) \\ &\vdots \\ &= f(f(a_1, a_2, w_3^{n-1}), f(w_n, y_2^n), \dots, f(a_n, z_2^n)), \end{aligned}$$

where $w_i = f(\overset{(n-3)}{a_2}, \bar{a}_2, x_i, a_2) \in H$.

Repeating this procedure for a_3, a_4 and so on, we obtain

$$f(b_1^n) = f(f(a_1^n), h_2^n).$$

This means that the operation f_H is well-defined.

It is easy to verify that $(G/H, f_H)$ is an n -ary group. Using the above procedure it is not difficult to see that H is the identity of G/H . Hence an n -ary group G/H is derived from the group $\text{Ret}_H(G/H)$. \square

Now, we return to the representations, again. Consider a representation $\Lambda : (G, f) \rightarrow GL(V)$. It is easy to see that $\ker \Lambda$ is a normal subgroup of G . Let H be a normal n -ary subgroup of (G, f) such that $H \subseteq \ker \Lambda$. Then, there is a representation $\bar{\Lambda} : G/H \rightarrow GL(V)$ such that

$$\bar{\Lambda}(aH) = \Lambda(a).$$

Conversely, from every representation of G/H , we obtain a representation of G . On the other hand, G/H is of reduced type, and hence its representations are the same as the ordinary representations of $\text{Ret}_H(G/H)$. So, we proved,

Proposition 5.6. *There is a bijection between ordinary representations of $\text{Ret}_H(G/H)$ and the set of representations of G with the property $H \subseteq \ker \Lambda$.*

Proposition 5.7. *A simple n -ary group which is not strongly simple is b -derived from an abelian group or it is reducible to a non-abelian group.*

Proof. Suppose $H = \{p\}$ is a normal n -ary subgroup of (G, f) . Then we have

$$f(p, p, \dots, p) = p, \quad \bar{p} = p, \quad \forall x \in G : f(\overset{(n-3)}{x}, \bar{x}, p, x) = p.$$

Hence

$$\begin{aligned} f(p, x_2^n) &= f(f(\overset{(n-2)}{x_2}, \bar{x}_2, p), x_2^n) \\ &= f(x_2, f(\overset{(n-3)}{x_2}, \bar{x}_2, p, x_2), x_3^n) \\ &= f(x_2, p, x_3^n). \end{aligned}$$

This shows that p is a central element and, according to [8], an n -ary group (G, f) is b -derived from a binary group (G, \cdot) . Hence, $Z(G, f) = Z(G, \cdot)$ is a normal n -ary subgroup of (G, f) . But G has no proper normal subgroups, so there are two cases:

- (1) $Z(G, \cdot) = G$ and so (G, f) is b -derived from an abelian group,
- (2) $Z(G, \cdot)$ is singleton and hence $b = 1$. In this case (G, f) is reducible to a non-abelian group (G, \cdot) .

□

Remark 5.8. To find representations of an n -ary group (G, f) , we have four cases, as follow,

- (1) only $H = G$ is a normal subgroup of (G, f) , (in this case (G, f) has only trivial representation),
- (2) (G, f) is b -derived from an abelian group,
- (3) $(G, f) = \text{der}(G, \cdot)$, (in this case representations of (G, f) are the same as the representations of (G, \cdot)),
- (4) (G, f) has proper normal n -ary subgroups, (in this case, if we know the set of normal n -ary subgroups of (G, f) , then we obtain all its representations from representations of the groups $\text{Ret}_H(G/H)$).

Finally, summarizing results of this section, we have the following theorem:

Theorem 5.9. *Representation theory of n -ary groups, reduces to the following three problems,*

- a) *representations of b -derived ternary groups from abelian groups,*
- b) *determining all normal ternary subgroup,*
- c) *representation theory of ordinary groups.*

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REPRESENTATION THEORY OF POLYADIC GROUPS

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ABSTRACT. In this article, we introduce the notion of representations of polyadic groups and we investigate the connection between these representations and those of retract groups and covering groups.

1. INTRODUCTION

A non-empty set G together with an n -ary operation $f : G^n \rightarrow G$ is called an n -ary groupoid and is denoted by (G, f) . We will assume that $n > 2$.

According to the general convention used in the theory of n -ary systems, the sequence of elements x_i, x_{i+1}, \dots, x_j is denoted by x_i^j . In the case $j < i$ it is the empty symbol. If $x_{i+1} = x_{i+2} = \dots = x_{i+t} = x$, then instead of x_{i+1}^{i+t} we write $\overset{(t)}{x}$. In this convention $f(x_1, \dots, x_n) = f(x_1^n)$ and

$$f(x_1, \dots, x_i, \underbrace{x, \dots, x}_t, x_{i+t+1}, \dots, x_n) = f(x_1^i, \overset{(t)}{x}, x_{i+t+1}^n).$$

An n -ary groupoid (G, f) is called (i, j) -associative, if

$$(1.1) \quad f(x_1^{i-1}, f(x_i^{n+i-1}), x_{n+i}^{2n-1}) = f(x_1^{j-1}, f(x_j^{n+j-1}), x_{n+j}^{2n-1})$$

holds for all $x_1, \dots, x_{2n-1} \in G$. If this identity holds for all $1 \leq i < j \leq n$, then we say that the operation f is associative and (G, f) is called an n -ary semigroup.

If, for all $x_0, x_1, \dots, x_n \in G$ and fixed $i \in \{1, \dots, n\}$, there exists an element $z \in G$ such that

$$(1.2) \quad f(x_1^{i-1}, z, x_{i+1}^n) = x_0,$$

then we say that this equation is i -solvable or solvable at the place i . If this solution is unique, then we say that (1.2) is uniquely i -solvable.

An n -ary groupoid (G, f) uniquely solvable for all $i = 1, \dots, n$, is called an n -ary quasigroup. An associative n -ary quasigroup is called an n -ary group or a polyadic group. In the binary case (i.e., for $n = 2$) it is a usual group.

Now, such and similar n -ary systems have many applications in different branches. For example, in the theory of automata, (cf. [11]), n -ary

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semigroups and n -ary groups are used, some n -ary groupoids are applied in the theory of quantum groups (cf. [15]). Different applications of ternary structures in physics are described by R. Kerner (cf. [13]). In physics there are used also such structures as n -ary Filippov algebras (cf. [16]) and n -Lie algebras (cf. [18]).

The idea of investigations of such groups seems to be going back to E. Kasner's lecture [12] at the fifty-third annual meeting of the American Association for the Advancement of Science in 1904. But the first paper concerning the theory of n -ary groups was written (under inspiration of Emmy Noether) by W. Dörnte in 1928 (see [2]). In this paper Dörnte observed that any n -ary groupoid (G, f) of the form $f(x_1^n) = x_1 \circ x_2 \circ \dots \circ x_n \circ b$, where (G, \circ) is a group and b is its fixed element belonging to the center of (G, \circ) , is an n -ary group. Such n -ary groups, called b -derived from the group (G, \circ) , are denoted by $der_b(G, \circ)$. In the case when b is the identity of (G, \circ) we say that such n -ary group is *reducible* to the group (G, \circ) or *derived* from (G, \circ) . But for every $n > 2$ there are n -ary groups which are not derived from any group. An n -ary group (G, f) is derived from some group iff it contains an element e (called an *n -ary identity*) such that

$$(1.3) \quad f(\overset{(i-1)}{e}, x, \overset{(n-i)}{e}) = x$$

holds for all $x \in G$ and $i = 1, \dots, n$.

It is worthwhile to note that in the definition of an n -ary group, under the assumption of the associativity of the operation f , it suffices only to postulate the existence of a solution of (1.2) at the places $i = 1$ and $i = n$ or at one place i other than 1 and n (cf. [17], p. 213). Other useful characterizations of n -ary groups one can find in [3] and [6].

From the definition of an n -ary group (G, f) , we can directly see that for every $x \in G$, there exists only one $z \in G$ satisfying the equation

$$(1.4) \quad f(\overset{(n-1)}{x}, z) = x.$$

This element is called *skew* to x and is denoted by \bar{x} . In a ternary group ($n = 3$) derived from the binary group (G, \cdot) the skew element coincides with the inverse element in (G, \circ) . Thus, in some sense, the skew element is a generalization of the inverse element in binary groups. Dörnte proved (see [2]) that in ternary groups we have $f(x, y, z) = f(\bar{z}, \bar{y}, \bar{x})$ and $\bar{\bar{x}} = x$, but for $n > 3$ this is not true. For $n > 3$ there are n -ary groups in which one fixed element is skew to all elements (cf. [4]) and n -ary groups in which any element is skew to itself.

Nevertheless, the concept of skew elements plays a crucial role in the theory of n -ary groups. Namely, as Dörnte proved (see also [6]), the following theorem is true.

Theorem 1.1. *In any n -ary group (G, f) the following identities*

$$(1.5) \quad f(\overset{(i-2)}{x}, \bar{x}, \overset{(n-i)}{x}, y) = f(y, \overset{(n-j)}{x}, \bar{x}, \overset{(j-2)}{x}) = y,$$

$$(1.6) \quad f\left(\begin{smallmatrix} (k-1) \\ x \end{smallmatrix}, \bar{x}, \begin{smallmatrix} (n-k) \\ x \end{smallmatrix}\right) = x$$

hold for all $x, y \in G$, $2 \leq i, j \leq n$ and $1 \leq k \leq n$.

One can prove (cf. [3]) that for $n > 2$ an n -ary group can be defined as an algebra $(G, f, \bar{})$ with one associative n -ary operation f and one unary operation $\bar{}: x \rightarrow \bar{x}$ satisfying for some $2 \leq i, j \leq n$ the identities (1.5). This means that a non-empty subset H of an n -ary group (G, f) is its subgroup iff it is closed with respect to the operation f and $\bar{x} \in H$ for every $x \in H$.

Fixing in an n -ary operation f all inner elements a_2, \dots, a_{n-1} we obtain a new binary operation

$$x * y = f(x, a_2^{n-1}, y).$$

Such obtained groupoid $(G, *)$ is called a *retract* of (G, f) . Choosing different elements a_1, \dots, a_{n-1} we obtain different retracts. Retracts of n -ary groups are groups. Retracts of a fixed n -ary group are isomorphic (cf. [8]). So, we can consider only retracts of the form

$$x * y = f(x, \begin{smallmatrix} (n-2) \\ a \end{smallmatrix}, y).$$

Such retracts will be denoted by $Ret_a(G, f)$, or simply by $Ret_a(G)$. The identity of the group $Ret_a(G)$ is \bar{a} . One can verify that the inverse element to x has the form

$$(1.7) \quad x^{-1} = f(\bar{a}, \begin{smallmatrix} (n-3) \\ x \end{smallmatrix}, \bar{x}, \bar{a}).$$

Binary retracts of an n -ary group (G, f) are commutative only in the case when there exists an element $a \in G$ such that

$$f(x, \begin{smallmatrix} (n-2) \\ a \end{smallmatrix}, y) = f(y, \begin{smallmatrix} (n-2) \\ a \end{smallmatrix}, x)$$

holds for all $x, y \in G$. An n -ary group with this property is called *semiabelian*. It satisfies the identity

$$(1.8) \quad f(x_1^n) = f(x_n, x_2^{n-1}, x_1)$$

(cf. [3]).

One can prove (cf. [9]) that a semiabelian n -ary group is *medial*, i.e., it satisfies the identity

$$(1.9) \quad f(f(x_{11}^{1n}), f(x_{21}^{2n}), \dots, f(x_{n1}^{nn})) = f(f(x_{11}^{n1}), f(x_{12}^{n2}), \dots, f(x_{1n}^{nn})).$$

In such n -ary groups

$$(1.10) \quad \overline{f(x_1, x_2, x_3, \dots, x_n)} = f(\bar{x}_1, \bar{x}_2, \bar{x}_3, \dots, \bar{x}_n)$$

for all $x_1, \dots, x_n \in G$.

Any n -ary group can be uniquely described by its retract and some automorphism of this retract. Namely, the following Hosszú-Gluskin Theorem (cf. [5] or [7]) is valid.

Theorem 1.2. *An n -ary groupoid (G, f) is an n -ary group iff*

- (1) *on G one can define an operation \cdot such that (G, \cdot) is a group,*

- (2) *there exist an automorphism φ of (G, \cdot) and $b \in G$ such that $\varphi(b) = b$,*
- (3) *$\varphi^{n-1}(x) = b \cdot x \cdot b^{-1}$ for every $x \in G$,*
- (4) *$f(x_1^n) = x_1 \cdot \varphi(x_2) \cdot \varphi^2(x_3) \cdot \dots \cdot \varphi^{n-1}(x_n) \cdot b$ for all $x_1, \dots, x_n \in G$.*

One can prove that $(G, \cdot) = \text{Ret}_a(G, f)$ for some $a \in G$. In connection with this we say that an n -ary group (G, f) is (φ, b) -derived from the group (G, \cdot) .

The main aim of this article is to introduce *representations* of n -ary groups and to investigate their main properties, with a special focus on ternary groups. Note that, this is not the first attempt to study representations of n -ary groups, because there are some other articles, with different point of views concerning representations on n -ary groups, (cf. [1], [10], [17] and [19]). However, our method seems to be the most natural generalization of the notion of representation from binary to n -ary groups.

2. ACTION OF AN n -ARY GROUP ON A SET

Suppose that (G, f) is an n -ary group and A is a non-empty set. We say that (G, f) *acts* on A if for all $x \in G$ and $a \in A$ corresponds a unique element $x.a \in A$ such that

- (i) $f(x_1^n).a = x_1.(x_2.(x_3. \dots .(x_n.a)) \dots)$ for all $x_1, \dots, x_n \in G$,
- (ii) for all $a \in A$, there exists $x \in G$ such that $x.a = a$,
- (iii) the map $a \mapsto x.a$ is a bijection for all $x \in G$.

For $a \in A$, we define the *stabilizer* G_a of a as follows

$$G_a = \{x \in G : x.a = a\}.$$

Proposition 2.1. *G_a is an n -ary subgroup of (G, f) .*

Proof. By condition (ii) of the above definition G_a is non-empty. Since for $x_1, x_2, \dots, x_n \in G_a$ we have

$$f(x_1^n).a = x_1.(x_2.(x_3. \dots .(x_n.a)) \dots) = a,$$

$f(x_1^n) \in G_a$. Hence G_a is closed with respect to the operation f .

Now if $x \in G_a$, then by (1.6) we obtain

$$a = x.a = f(\bar{x}, \overset{(n-1)}{x}).a = \bar{x}.(x. \dots .x.(x.a)) \dots = \bar{x}.a,$$

which implies $\bar{x} \in G_a$. This completes the proof. \square

Proposition 2.2. *If an n -ary group (G, f) acts on a set A , then the relation \sim defined on A by*

$$a \sim b \iff \exists x \in G : x.a = b$$

is an equivalence relation.

Proof. For each $a \in A$ there is $x \in G$ such that $x.a = a$, so $a \sim a$. If $a \sim b$ for $a, b \in A$, then $z.a = b$ for some $z \in G$. Let y be the unique solution of the equation

$$f(y, z, \overset{(n-2)}{x}) = x,$$

where $x \in G$ is such that $x.a = a$. For this y we have $y.b = a$ since

$$a = x.a = f(y, z, \overset{(n-2)}{x}).a = y.z.a = y.b.$$

Thus $b \sim a$. Finally, let $a \sim b$ and $b \sim c$. Then there are $x, y, z \in G$ such that $x.a = b$, $y.b = c$ and $z.b = b$. In this case for $u = f(y, \overset{(n-2)}{z}, x)$ we have

$$u.a = f(y, \overset{(n-2)}{z}, x).a = y.b = c,$$

which proves $a \sim c$. \square

Theorem 2.3. *The formula $x.a = f(x, a, \overset{(n-3)}{x}, \bar{x})$ defines an action of an n -ary group G on itself.*

Proof. The last condition of Theorem 1.2 can be written in the form

$$f(x_1^n) = x_1 \cdot \varphi(x_2) \cdot \varphi^2(x_3) \cdot \dots \cdot \varphi^{n-2}(x_{n-1}) \cdot b \cdot x_n.$$

Thus $\bar{x} = (\varphi(x) \cdot \varphi^2(x) \cdot \dots \cdot \varphi^{n-2}(x) \cdot b)^{-1}$. Consequently

$$(2.1) \quad x.a = x \cdot \varphi(a) \cdot \varphi(x^{-1}).$$

Hence

$$\begin{aligned} y.(x.a) &= y \cdot \varphi(x) \cdot \varphi^2(a) \cdot \varphi^2(x^{-1}) \cdot \varphi(y)^{-1} \\ &= y \cdot \varphi(x) \cdot \varphi^2(a) \cdot \varphi((y \cdot \varphi(x))^{-1}). \end{aligned}$$

Iterating this procedure we obtain

$$\begin{aligned} &x_1.(x_2.(x_3 \dots (x_n.a)) \dots) = \\ &x_1 \cdot \varphi(x_2) \cdot \varphi^2(x_3) \cdot \dots \cdot \varphi^{n-1}(x_n) \cdot \varphi^n(a) \cdot \varphi((x_1 \cdot \varphi(x_2) \cdot \varphi^2(x_3) \cdot \dots \cdot \varphi^{n-1}(x_n))^{-1}). \end{aligned}$$

Since $\varphi^n(a) = b \cdot \varphi(a) \cdot b^{-1}$ from the above we obtain

$$x_1.(x_2.(x_3 \dots (x_n.a)) \dots) = f(x_1^n) \cdot \varphi(a) \cdot \varphi(f(x_1^n)^{-1}).$$

This by (2.1) gives $f(x_1^n).a = x_1.(x_2.(x_3 \dots (x_n.a)) \dots)$. \square

Proposition 2.4. *In semiabelian n -ary groups the relation*

$$a \sim b \iff \exists x \in G : f(x, a, \overset{(n-3)}{x}, \bar{x}) = b$$

is a congruence.

Proof. Indeed, by Proposition 2.2 it is an equivalence relation. To prove that it is a congruence let $a_i \sim b_i$, i.e., $f(x_i, a_i, \overset{(n-3)}{x_i}, \bar{x}_i) = b_i$ for some $x_i \in G$ and all $i = 1, \dots, n$. Then

$$f(b_1^n) = f(f(x_1, a_1, \overset{(n-3)}{x_1}, \bar{x}_1), f(x_2, a_2, \overset{(n-3)}{x_2}, \bar{x}_2), \dots, f(x_n, a_n, \overset{(n-3)}{x_n}, \bar{x}_n)),$$

which by the mediality and (1.10) gives

$$f(b_1^n) = f(f(x_1^n), f(a_1^n), \underbrace{f(x_1^n), \dots, f(x_1^n)}_{n-3}, \overline{f(x_1^n)}).$$

Thus $f(a_1^n) \sim f(b_1^n)$. □

Remark 2.5. The formula (2.1) says that in n -ary groups b -derived from a group (G, \cdot) the above relation coincides with the conjugation in (G, \cdot) . Thus in non-semiabelian n -ary groups it may not be a congruence.

Elements belonging to the same equivalence class are called *conjugate*. The equivalence classes are called *conjugate classes* of an n -ary group G and have the form

$$Cl_G(a) = \{f(x, a, \overset{(n-3)}{x}, \overline{x}) : x \in G\}.$$

As a simple consequence of (1.9) and (1.10) we obtain

Proposition 2.6. *In semiabelian n -ary group the set containing all elements of G conjugated with elements of a given n -ary subgroup also is an n -ary subgroup.*

For $a \in G$, we define the *centralizer* of a , as follows

$$C_G(a) = \{x \in G : f(x, a, \overset{(n-3)}{x}, \overline{x}) = a\}.$$

From Theorem 1.1 it follows that in n -ary groups b -derived from a group (G, \cdot) the centralizer of any $a \in G$ coincides with the centralizer of a in (G, \cdot) .

Proposition 2.7. *For every $x \in C_G(a)$ and every $0 \leq i, j, k \leq n-2$ such that $i + j + k = n-2$ we have*

$$f(\overset{(i)}{x}, a, \overset{(j)}{x}, \overline{x}, \overset{(k)}{x}) = f(\overset{(i)}{x}, \overline{x}, \overset{(j)}{x}, a, \overset{(k)}{x}) = a.$$

Proof. For every $x \in C_G(a)$, we have $f(x, a, \overset{(n-3)}{x}, \overline{x}) = a$. Multiplying this equation on the left by x and on the right by $x, \dots, x, \overline{x}$ ($n-2$ elements), we obtain

$$f(x, f(x, a, \overset{(n-3)}{x}, \overline{x}), \overset{(n-3)}{x}, \overline{x}) = f(x, a, \overset{(n-3)}{x}, \overline{x}) = a,$$

which in view of the associativity of the operation f and (1.6) gives

$$f(x, x, a, \overset{(n-4)}{x}, \overline{x}) = a.$$

Repeating this procedure we obtain

$$f(\overset{(i)}{x}, a, \overset{(n-i-2)}{x}, \overline{x}) = a$$

for every $1 \leq i \leq n-2$. Theorem 1.1 completes the proof. □

3. G-MODULES AND REPRESENTATIONS

All vector spaces in this section are defined over the field of complex numbers and have finite dimension.

Definition 3.1. Suppose that an n -ary group G acts on a vector space V and we have

- (1) $x.(\lambda v + u) = \lambda x.v + x.u,$
- (2) $\exists p \in G \forall v \in V : p.v = v.$

Then we call (V, p) , or simply V , a G -module.

Notions, such as G -submodule, G -homomorphism, irreducibility and so on, are defined by the ordinary way.

Definition 3.2. A map $\Lambda : G \rightarrow GL(V)$ with the property

$$\Lambda(f(x_1^n)) = \Lambda(x_1)\Lambda(x_2) \dots \Lambda(x_n)$$

is a *representation* of G , provided that $\ker \Lambda$ is non-empty. The function

$$\chi(x) = \text{Tr } \Lambda(x)$$

is called the corresponding *character* of Λ .

Remark 3.3. If V is a G -module, then Λ defined by

$$\Lambda(x)(v) = x.v$$

is a representation of G . The converse is also true.

Example 3.4. Let A be an arbitrary binary group with a normal subgroup H . Let $a \in A \setminus H$ be an involution. Then $G = aH$ with the operation

$$f(x, y, z) = xyz$$

is a ternary group. If Λ is an ordinary representation of A with the property $a \in \ker \Lambda$, then, clearly Λ is also a representation of G . For example, suppose $A = GL_n(\mathbb{C})$ and $H = SL_n(\mathbb{C})$. Let $a = \text{diag}(-1, 1, \dots, 1)$ and define $G = aH$. Then, every representation of A in which $a \in \ker \Lambda$ is also a representation of a ternary group G .

Example 3.5. For any subgroup H of an ordinary group A and any element $a \in Z(A) \setminus H$ with the order n we define on $G = aH$ an n -ary operation by

$$f(x_1, x_2, \dots, x_n) = ax_1x_2 \dots x_n.$$

This operation is associative, because $a \in Z(A)$. Also, G is closed under this operation, since $o(a) = n$. So, G is an n -ary group. Any A -representation Λ with $a \in \ker \Lambda$ is also a G -representation.

Example 3.6. The set $G = \mathbb{Z}_n$ with the ternary operation

$$f(x, y, z) = x - y + z \pmod{n}$$

is, by Theorem 1.2, a ternary group. We want to classify all representations of G .

Let $\Lambda : G \rightarrow GL_m(\mathbb{C})$ be any representation. Then we have

$$\Lambda(f(x, y, z)) = \Lambda(x)\Lambda(y)\Lambda(z),$$

equivalently,

$$\Lambda(x - y + z) = \Lambda(x)\Lambda(y)\Lambda(z).$$

We have

$$\Lambda(x + y) = \Lambda(x)\Lambda(0)\Lambda(y), \quad \Lambda(x - y) = \Lambda(x)\Lambda(y)\Lambda(0).$$

Suppose $A = \Lambda(0)$. We have

$$\Lambda(x + y) = \Lambda(x)A\Lambda(y).$$

It is easy to see that $A^2 = I$. Now, define $\Lambda'(x) = A\Lambda(x)$. Then

$$\Lambda'(x + y) = \Lambda'(x)\Lambda'(y),$$

and so, Λ' is an ordinary representation of $(\mathbb{Z}_n, +)$. Hence, every representation of the ternary group G is of the form $\Lambda(x) = A\Lambda'(x)$, where A is an involution and Λ' is an ordinary representation of $(\mathbb{Z}_n, +)$.

Similarly, we can classify all representations of ternary groups of the form $G = (A, f)$, where A is an ordinary abelian group and

$$f(x, y, z) = x - y + z.$$

Theorem 3.7. (Maschke) *Let G be a finite n -ary group. Then every G -module is completely reducible.*

Proof. Let (V, p) be a G -module and $W \leq_G V$. Suppose $V = W \oplus X$, where X is just a subspace. Let $\varphi : V \rightarrow W$ be the corresponding projection. Define a new map $\theta : V \rightarrow V$ as

$$\theta(v) = \frac{1}{|G|} \sum_{x \in G} \bar{x} \cdot \varphi(x.v).$$

It is easy to see that

$$\theta(x.v) = x.p. \dots .p.\theta(v) = x.\theta(v).$$

So θ is a G -homomorphism and hence its kernel is a G -submodule. For all $w \in W$, we have $\theta(w) = w$ and so $\theta^2 = \theta$. Now, we have $V = W \oplus \ker \theta$. \square

Remark 3.8. Any G -module (V, p) is also an ordinary $Ret_p(G)$ -module, because

$$(x * y).v = f(x, \overset{(n-2)}{p}, y).v = x.p. \dots .p.y.v = x.y.v.$$

From now on, we will assume that $e \in G$ is an arbitrary fixed element. For all $p \in G$, we have $Ret_e(G) \cong Ret_p(G)$ and further the isomorphism is given by the following rule

$$h(x) = f(\overset{(n-2)}{e}, x, \bar{p}).$$

By \hat{G} we denote the binary group $Ret_e(G)$. If (V, p) is a G -module, then we can define a \hat{G} -module structure on V by $x \circ v = h(x).v$. So, we have

$$x \circ v = f\left(\begin{smallmatrix} (n-2) \\ e \end{smallmatrix}, x, \bar{p}\right).v = e. \dots .e.x.\bar{p}.v.$$

But, we have $\bar{p}.v = \bar{p}.p. \dots .p.v = f\left(\bar{p}, \begin{smallmatrix} (n-1) \\ p \end{smallmatrix}\right).v = p.v = v$. Hence

$$x \circ v = \underbrace{e. \dots .e}_{n-2}.x.v.$$

Now, every G -module is also a \hat{G} -module, but the converse is not true in general. During this article, we will give some necessary and sufficient conditions for a \hat{G} -module to be also a G -module. The next proposition is the first condition of this type.

Proposition 3.9. *Let V be a \hat{G} -module. Then V is a G -module iff*

$$\forall x_2, \dots, x_{n-1} \in G \ \forall v : \ f(\bar{e}, x_2^{n-1}, \bar{e}).v = x_2.x_3. \dots .x_{n-1}.v.$$

Proof. We have

$$\begin{aligned} f(x_1^n) &= f(f(x_1, \begin{smallmatrix} (n-2) \\ e \end{smallmatrix}, \bar{e}), x_2^n) \\ &= f(x_1, \begin{smallmatrix} (n-2) \\ e \end{smallmatrix}, f(\bar{e}, x_2^n)) \\ &= x_1 * f(\bar{e}, x_2^n) \\ &= x_1 * f(\bar{e}, x_2^{n-1}, f(\bar{e}, \begin{smallmatrix} (n-2) \\ e \end{smallmatrix}, x_n)) \\ &= x_1 * f(\bar{e}, x_2^{n-1}, \bar{e}) * x_n. \end{aligned}$$

So, the equality

$$f(x_1^n).v = x_1.x_2. \dots .x_{n-1}.x_n.v$$

holds, iff

$$f(\bar{e}, x_2^{n-1}, \bar{e}).v = x_2.x_3. \dots .x_{n-1}.v$$

for all x_2, \dots, x_{n-1} and v . □

Remark 3.10. Suppose that V is a G -module in which the corresponding representation is Λ . We know that V is also a \hat{G} -module. The corresponding representation of this last module is

$$\hat{\Lambda}(x) = \underbrace{\Lambda(e) \dots \Lambda(e)}_{n-2} \Lambda(x).$$

Because in \hat{G} , the identity element is \bar{e} , we have

$$\hat{\Lambda}(\bar{e}) = id.$$

So $\Lambda(e)^{n-2} \Lambda(\bar{e}) = id$ and hence

$$\Lambda(\bar{e}) = \Lambda(e)^{2-n}.$$

In the sequel, the corresponding character of $\hat{\Lambda}$, will be denoted by $\hat{\chi}$.

Proposition 3.11. *Suppose that Λ is a representation of G with the character χ . Then χ is fixed on the conjugate classes of G .*

Proof. Indeed, for every $b \in Cl_G(a)$ we have

$$\Lambda(b) = \Lambda(f(x, a, \binom{n-3}{x}, \bar{x})) = \Lambda(x)\Lambda(a)\Lambda(x)^{n-3}\Lambda(\bar{x}),$$

so

$$\begin{aligned} \chi(b) &= Tr(\Lambda(x)\Lambda(a)\Lambda(x)^{n-3}\Lambda(\bar{x})) \\ &= Tr(\Lambda(x)\Lambda(a)\Lambda(e)^{n-2}\Lambda(\bar{e})\Lambda(x)^{n-3}\Lambda(\bar{x})) \\ &= Tr(\Lambda(a)\Lambda(e)^{n-2}\Lambda(\bar{e})\Lambda(x)^{n-3}\Lambda(\bar{x})\Lambda(x)) \\ &= Tr(\Lambda(a)\Lambda(e)^{n-2}\Lambda(f(\bar{e}, \binom{n-3}{x}, \bar{x}, x))) \\ &= Tr(\Lambda(a)\Lambda(e)^{n-2}\Lambda(\bar{e})) \\ &= Tr \Lambda(a) \\ &= \chi(a). \end{aligned}$$

This completes the proof. \square

Proposition 3.12. *Suppose that $\Lambda : (G, f) \rightarrow GL(V)$ is a representation of the finite n -ary group (G, f) with the corresponding character χ . Let*

$$\ker \chi = \{x \in G : \chi(x) = \dim V\}.$$

Then $\ker \chi = \ker \Lambda$.

Proof. Let $\dim V = m$. It is clear that $\ker \Lambda \subseteq \ker \chi$. Moreover, for each $x \in G$ of order k we have

$$\Lambda(x)^{m^k} = \Lambda(x).$$

Hence $\Lambda(x)$ is a root of the polynomial $T^{m^k-1} - 1$. But, this polynomial has distinct roots in \mathbb{C} , so $\Lambda(x)$ can be diagonalized, i.e.,

$$\Lambda(x) \sim \text{diag}(\varepsilon_1, \dots, \varepsilon_m),$$

where all ε_i are roots of unity. Now, we have

$$\chi(x) = \varepsilon_1 + \dots + \varepsilon_m.$$

If $\chi(x) = m$, then $\varepsilon_i = 1$ for all i . Hence $\Lambda(x) = id$ and so $x \in \ker \Lambda$. This completes the proof. \square

In the next proposition, we obtain the explicit form of the character $\hat{\chi}$.

Proposition 3.13. *Let χ be a character of an n -ary group (G, f) . Then for any $p \in \ker \chi$ we have*

$$\hat{\chi}(x) = \chi(f(\binom{n-2}{e}, x, \bar{p})).$$

Proof. We know that χ is a character of $Ret_p(G)$. On the other hand there is an isomorphism

$$h : Ret_e(G) \rightarrow Ret_p(G),$$

where $h(x) = f(\binom{n-2}{e}, x, \bar{p})$. So, the composite map $\chi \circ h$ is a character of $Ret_e(G)$. Let Λ be the corresponding representation of χ . Now, we have

$$\begin{aligned} \chi(h(x)) &= Tr(\Lambda(e)^{n-2} \Lambda(x) \Lambda(\bar{p})) \\ &= Tr(\Lambda(e)^{n-2} \Lambda(x)) \\ &= Tr \hat{\Lambda}(x). \end{aligned}$$

Hence $\hat{\chi}(x) = \chi(f(\binom{n-2}{e}, x, \bar{p}))$. \square

Remark 3.14. Now, for any irreducible character χ of an n -ary group (G, f) , we have an ordinary irreducible character $\hat{\chi}$ of the binary group $\hat{G} = Ret_e(G)$. So, we obtain the following orthogonality relation for the irreducible characters of G :

$$\frac{1}{|G|} \sum_{x \in G} \chi_1(f(\binom{n-2}{e}, x, \bar{p}_1)) \overline{\chi_2(f(\binom{n-2}{e}, x, \bar{p}_2))} = \delta_{\hat{\chi}_1, \hat{\chi}_2},$$

where $p_1 \in \ker \chi_1$ and $p_2 \in \ker \chi_2$ are arbitrary elements.

Proposition 3.15. *If a representation $\Gamma : Ret_e(G, f) \rightarrow GL(V)$ is also a representation of the n -ary group (G, f) , then*

$$\Gamma(\bar{x}) = \Gamma(x)^{2-n}$$

for every $x \in G$.

Proof. Indeed, $f(\binom{n-1}{x}, \bar{x}) = x$ implies $\Gamma(x)^{n-1} \Gamma(\bar{x}) = \Gamma(x)$, which gives $\Gamma(\bar{x}) = \Gamma(x)^{2-n}$. \square

Corollary 3.16. *Let (G, f) be a ternary group. Then a representation $\Gamma : Ret_e(G, f) \rightarrow GL(V)$ is also a representation of (G, f) iff*

$$\Gamma(\bar{x}) = \Gamma(x)^{-1}$$

for every $x \in G$.

Proof. From Proposition 3.9 it follows that $\Gamma : Ret_e(G, f) \rightarrow GL(V)$ is a representation of a ternary group (G, f) iff it satisfies the identity

$$\Gamma(f(\bar{e}, x, \bar{e})) = \Gamma(x).$$

If $\Gamma(\bar{x}) = \Gamma(x)^{-1}$ holds for all $x \in G$, then, in view of (1.7), for all $x \in G$ we have

$$\Gamma(f(\bar{e}, x, \bar{e})) = \Gamma(f(\bar{e}, \bar{\bar{x}}, \bar{e})) = \Gamma(\bar{x}^{-1}) = \Gamma(\bar{x})^{-1} = \Gamma(x).$$

Hence Γ is a representation of (G, f) .

The converse statement is a consequence of Proposition 3.15. \square

Remark 3.17. We can use the above proposition to obtain some deeper results in the case when G has a central element. Note that, according to [8], an n -ary group (G, f) has a central element iff it is b -derived from a binary group (G, \cdot) and $b \in Z(G, \cdot)$. Obviously, in this case $Z(G, f) = Z(G, \cdot)$.

Proposition 3.18. *Let e be a central element of an n -ary group $(G, f) = \text{der}_b(G, \cdot)$. Then a representation $\Gamma : \text{Ret}_e(G) \rightarrow GL(V)$ is a representation of (G, f) iff*

$$\Gamma(x_2x_3 \dots x_n e^{2-n}) = \Gamma(x_2)\Gamma(x_3) \dots \Gamma(x_n)$$

for all $x_2, \dots, x_n \in G$.

Proof. Since $(G, f) = \text{der}_b(G, \cdot)$ the binary operation in $\text{Ret}_e(G, f)$ has the form

$$x * y = f(x, \overset{(n-2)}{e}, y) = xy e^{n-2} b.$$

For a representation Γ of $\text{Ret}_e(G, f)$, we have

$$(3.1) \quad \Gamma(x * y) = \Gamma(x)\Gamma(y).$$

Now, for Γ to be a representation of (G, f) , it is necessary and sufficient that

$$\Gamma(f(x_1^n)) = \Gamma(x_1x_2 \dots x_nb) = \Gamma(x_1)\Gamma(x_2) \dots \Gamma(x_n).$$

If we replace in (3.1), y by $x_2 \dots x_n e^{2-n}$, we obtain

$$\Gamma(x_1x_2 \dots x_nb) = \Gamma(x_1)\Gamma(x_2 \dots x_n e^{2-n}).$$

So Γ is a representation of (G, f) , iff

$$\Gamma(x_2x_3 \dots x_n e^{2-n}) = \Gamma(x_2)\Gamma(x_3) \dots \Gamma(x_n)$$

for all $x_2, \dots, x_n \in G$. □

In an n -ary group $(G, f) = \text{der}_b(G, \cdot)$ we have $\bar{x} = x^{2-n}b^{-1}$. Hence, comparing the above result with Proposition 3.15 we obtain

Corollary 3.19. *Let e be a central element of an n -ary group $(G, f) = \text{der}_b(G, \cdot)$. If a representation $\Gamma : \text{Ret}_e(G) \rightarrow GL(V)$ is a representation of (G, f) , then $\Gamma(x^{2-n}b^{-1}) = \Gamma(x)^{2-n}$ for every $x \in G$.*

In the case of ternary groups, by Corollary 3.16, we obtain stronger result.

Corollary 3.20. *Let $(G, f) = \text{der}_b(G, \cdot)$ be a ternary group. Then a representation $\Gamma : \text{Ret}_e(G, f) \rightarrow GL(V)$ is also a representation of (G, f) , iff $\Gamma((bx)^{-1}) = \Gamma(x)^{-1}$ for every $x \in G$.*

Proposition 3.21. *Let e be a central element of an ternary group $(G, f) = \text{der}_b(G, \cdot)$. Then a character χ of $\text{Ret}_e(G, f)$ is a character of (G, f) iff for all $x \in G$ we have $\chi(\bar{x}) = \overline{\chi(x)}$.*

Proof. Let $\Gamma : \text{Ret}_e(G, f) \rightarrow GL(V)$ be a representation corresponding to χ . If χ is a character of (G, f) , then Γ is also a representation of (G, f) and so $\Gamma(\bar{x}) = \Gamma(x)^{-1}$. Hence we have $\chi(\bar{x}) = \overline{\chi(x)}$.

Conversely, if $\chi(\bar{x}) = \overline{\chi(x)}$ holds for all $x \in G$, then in particular $\overline{\chi(e)} = \chi(\bar{e})$. Thus $\chi(e) = \chi(\bar{e})$ because $\chi(\bar{e})$ is real. Now, for all $x \in G$, we have $x * \bar{x} = f(x, e, \bar{x}) = f(e, x, \bar{x}) = e$, so $\chi(x * \bar{x}) = \chi(e) = \chi(\bar{e})$. Hence,

$$x * \bar{x} \in \ker \chi = \ker \Gamma.$$

This shows that $\Gamma(x^{-1}) = \Gamma(\bar{x})$ and so Γ is a representation of G . Hence χ is also a character of G . \square

Proposition 3.22. *Let e be a central element of a ternary group $(G, f) = \text{der}_b(G, \cdot)$. If χ is a common character of (G, f) and $\text{Ret}_e(G, f)$, then $\hat{\chi} = \chi$.*

Proof. We have $\chi(\bar{e}) = \overline{\chi(e)}$, so $\chi(e)$ is real, and hence $\chi(e) = \chi(\bar{e})$. So $e \in \ker \chi$. Now, suppose $p = e$. Then

$$\hat{\chi}(x) = \chi(f(e, x, \bar{p})) = \chi(f(e, x, \bar{e})) = \chi(f(x, e, \bar{e})) = \chi(x),$$

which completes the proof. \square

In the remaining part of this section, we try to answer the problem: when $\hat{\Lambda}_1 \sim \hat{\Lambda}_2$? We give an answer to this question for n -ary groups with some central elements.

Proposition 3.23. *For an n -ary group (G, f) with a central element e the following assertions are true:*

- (1) *Let (V, p) be a G -module and $h : V \rightarrow V$ be a \hat{G} -homomorphism. Then h is also a G -homomorphism.*
- (2) *Let (V_1, p_1) and (V_2, p_2) be two G -modules and $h : V_1 \rightarrow V_2$ be a \hat{G} -homomorphism. Then h is a G -homomorphism, iff $h(e.v) = e.h(v)$.*
- (3) *Let (V_1, p_1) and (V_2, p_2) be two G -modules and $h : V_1 \rightarrow V_2$ be a \hat{G} -homomorphism. Then h is a G -homomorphism, iff $p_1.h(v) = h(v)$ for every $v \in V_1$.*
- (4) *Let (V_1, p_1) and (V_2, p_2) be two G -modules and*

$$V_1 \cong_{\hat{G}} V_2.$$

Then $V_1 \cong_G V_2$, iff for all $u \in V_2$, $p_1.u = u$.

Proof. (1). In view of $x * y = f(x, \overset{(n-2)}{e}, y)$, for a G -module (V, p) , we have

$$\begin{aligned}
 h(e.v) &= h(f(\overset{(n-1)}{e}, \bar{e}).v) \\
 &= h(f(f(\overset{(n-1)}{e}, \bar{e}), \overset{(n-1)}{p}).v) \\
 &= h(f(f(e, \overset{(n-2)}{p}, \bar{e}), \overset{(n-2)}{e}, p).v) \\
 &= h(f(e, \overset{(n-2)}{p}, \bar{e}) \circ v) \\
 &= f(e, \overset{(n-2)}{p}, \bar{e}) \circ h(v) \\
 &= f(f(e, \overset{(n-2)}{p}, \bar{e}), \overset{(n-2)}{e}, p).h(v) \\
 &= f(e, \overset{(n-2)}{p}, f(\bar{e}, \overset{(n-2)}{e}, p)).h(v) \\
 &= f(e, \overset{(n-1)}{p}).h(v) \\
 &= e.p. \dots .p.h(v) \\
 &= e.h(v).
 \end{aligned}$$

Now for all $x \in G$, we have $h(x \circ v) = x \circ h(v)$, so

$$h(\underbrace{e. \dots .e}_{n-2}.x.v) = e. \dots .e.x.h(v).$$

Hence

$$\underbrace{e. \dots .e}_{n-2}.h(x.v) = e. \dots .e.x.h(v).$$

Since the map $u \mapsto e.u$ is bijection, we have $h(x.v) = x.h(v)$.

(2). The proof of this part is just as the above.

(3). Suppose h is a G -homomorphism. Then $p_1.h(v) = h(p_1.v) = h(v)$ for every $v \in V_1$.

Conversely, assume that for all $v \in V_1$ holds $p_1.h(v) = h(v)$. Then

$$\begin{aligned}
 h(e.v) &= h(f(\overset{(n-1)}{e}, \bar{e}). \underbrace{p_1 \dots p_1}_{n-2}.v) \\
 &= h(\underbrace{e \dots e}_{n-1}. \bar{e}. \underbrace{p_1 \dots p_1}_{n-2}.v) \\
 &= h(f(e, \overset{(n-2)}{p_1}, \bar{e}). \underbrace{e \dots e}_{n-2}.v) \\
 &= h(f(e, \overset{(n-2)}{p_1}, \bar{e}) \circ v) \\
 &= f(e, \overset{(n-2)}{p_1}, \bar{e}) \circ h(v) \\
 &= f(e, \overset{(n-2)}{p_1}, \bar{e}). \underbrace{e \dots e}_{n-2}.h(v) \\
 &= f(e, \overset{(n-2)}{p_1}, \bar{e}). \underbrace{e \dots e}_{n-2}.p_1.h(v) \\
 &= f(f(e, \overset{(n-2)}{p_1}, \bar{e}), \overset{(n-2)}{e}, p_1).h(v) \\
 &= f(e, \overset{(n-2)}{p_1}, f(\bar{e}, \overset{(n-2)}{e}, p_1)).h(v) \\
 &= f(e, \overset{(n-1)}{p_1}).h(v) \\
 &= e.h(v).
 \end{aligned}$$

(4). Let $h : V_1 \rightarrow V_2$ be a G -isomorphism. Then h is also a \hat{G} -homomorphism, and hence $p_1.h = h$. Because h is onto, we obtain $p_1.u = u$, for all $u \in V_2$.

Conversely, suppose $p_1.u = u$, for all $u \in V_2$. Let $h : V_1 \rightarrow V_2$ be a \hat{G} -isomorphism. Then $p_1.h = h$, and so h is a G -isomorphism. \square

Proposition 3.24. *Let (G, f) be an n -ary group with a central element and let $\Lambda_1, \Lambda_2 : G \rightarrow GL(V)$ be two representations of (G, f) , such that $\hat{\Lambda}_1 \sim \hat{\Lambda}_2$. Then $\Lambda_1 \sim \Lambda_2$, iff $\ker \Lambda_1 = \ker \Lambda_2$.*

Proof. Let $p \in \ker \Lambda_1 = \ker \Lambda_2$. We define two G -modules V_1 and V_2 , as follows: V_1 is the vector space V with the action $x.v = \Lambda_1(x)(v)$, V_2 is the vector space V with the action $x.v = \Lambda_2(x)(v)$. Then $\hat{\Lambda}_1 \sim \hat{\Lambda}_2$ implies

$$V_1 \cong_{\hat{G}} V_2,$$

and $p.u = u$, for all $u \in V_2$. So, $V_1 \cong_G V_2$. This proves $\Lambda_1 \sim \Lambda_2$.

Conversely, let $\Lambda_1 \sim \Lambda_2$. Hence, we have $V_1 \cong_G V_2$. By the previous proposition, for $p \in \ker \Lambda_1$ and $u \in V_2$, we have $p.u = u$. Thus $\Lambda_2(p) = id$. Therefore, $\ker \Lambda_1 = \ker \Lambda_2$. \square

Corollary 3.25. *Let Λ_1 and Λ_2 be two representations of an n -ary group (G, f) with a central element e . If $\hat{\Lambda}_1 \sim \hat{\Lambda}_2$, then $\Lambda_1 \sim \Lambda_2$ iff $\Lambda_1(\bar{e}) \sim \Lambda_2(\bar{e})$.*

Proof. By the above proposition, $\Lambda_1 \sim \Lambda_2$, iff $\ker \Lambda_1 = \ker \Lambda_2$. But, we have

$$\ker \Lambda_1 = \{x \in G : \hat{\Lambda}_1(x) = \Lambda_1(e)^{n-2}\},$$

$$\ker \Lambda_2 = \{x \in G : \hat{\Lambda}_2(x) = \Lambda_2(e)^{n-2}\}.$$

Hence $\Lambda_1 \sim \Lambda_2$, iff $\Lambda_1(e)^{n-2} \sim \Lambda_2(e)^{n-2}$. But we have $\Lambda_1(e)^{n-2} = \Lambda_1(\bar{e})^{-1}$ and similarly for Λ_2 . So $\Lambda_1 \sim \Lambda_2$, iff $\Lambda_1(\bar{e})^{-1} \sim \Lambda_2(\bar{e})^{-1}$. \square

Remark 3.26. In the last two propositions and Corollary 3.25 the assumption that e is a central element can be replaced by the assumption that that an n -ary group (G, f) is semiabelian.

4. CONNECTION WITH THE REPRESENTATIONS OF THE COVERING GROUP

According to Post's Coset Theorem (cf. [17] or [14]) for any n -ary group (G, f) there exists a binary group (G^*, \cdot) and its normal subgroup H such that $G^*/H \simeq \mathbb{Z}_{n-1}$ and $G \subseteq G^*$ and

$$f(x_1^n) = x_1 \cdot x_2 \cdot x_3 \cdot \dots \cdot x_n$$

for all $x_1, \dots, x_n \in G$.

The group (G^*, \cdot) is called the *covering group* for (G, f) . We know several methods of a construction of such group. The smallest covering group has the form $G_a^* = G \times \mathbb{Z}_{n-1}$, where

$$\langle x, r \rangle \cdot \langle y, s \rangle = \langle f_*(x, \overset{(r)}{a}, y, \overset{(s)}{a}, \bar{a}, \overset{(n-2-r \diamond s)}{a}) \rangle, r \diamond s \rangle,$$

$r \diamond s = (r + s + 1)(\text{mod } (n-1))$ and $a \in G$ an arbitrary but fixed element. The symbol f_* means that the operation f is used one or two times (depending on the value s and t). Clearly fixing various element a of G , we obtain various groups but all these groups are isomorphic (cf. [14]).

The element $(\bar{a}, n-2)$ is the identity of the group (G_a^*, \cdot) . The inverse element has the form

$$\langle x, t \rangle^{-1} = \langle f_*(\bar{a}, \overset{(n-2-t)}{a}, \bar{x}, \overset{(n-3)}{x}, \bar{a}, \overset{(t+1)}{a}) \rangle, k \rangle,$$

where $k = (n-3-t)(\text{mod } (n-1))$.

The set G is identified with the subset $\{\langle x, 0 \rangle : x \in G\}$. Every retract of (G, f) is isomorphic to the normal subgroup

$$H = \{\langle x, n-2 \rangle : x \in G\}.$$

Suppose that V is a G_a^* -module. Then for $x_1, \dots, x_n \in G$ we have

$$\begin{aligned}
 x_1.x_2.x_3. \dots .x_n.v &= \langle x_1, 0 \rangle . \langle x_2, 0 \rangle . \langle x_3, 0 \rangle . \dots . \langle x_n, 0 \rangle . v \\
 &= \langle f(x_1, x_2, \bar{a}, \overset{(n-3)}{a}), 1 \rangle . \langle x_3, 0 \rangle . \dots . \langle x_n, 0 \rangle . v \\
 &= \langle f(f(x_1^2, \bar{a}, \overset{(n-3)}{a}), a, x_3, \bar{a}, \overset{(n-4)}{a}), 2 \rangle . \dots . \langle x_n, 0 \rangle . v \\
 &= \langle f(x_1^2, f(\bar{a}, \overset{(n-2)}{a}, x_3), \bar{a}, \overset{(n-4)}{a}), 2 \rangle . \dots . \langle x_n, 0 \rangle . v \\
 &= \langle f(x_1^3, \bar{a}, \overset{(n-4)}{a}), 2 \rangle . \dots . \langle x_n, 0 \rangle . v \\
 &\vdots \\
 &= \langle f(x_1^n), 0 \rangle . v \\
 &= f(x_1^n).v
 \end{aligned}$$

So, we obtain

Proposition 4.1. *Let (G_a^*, \cdot) be the covering group for an n -ary group (G, f) . Then for a G_a^* -module V to be a G -module it is necessary and sufficient that*

$$\exists p \in G \forall v \in V : p.v = v.$$

Hence, we proved

Proposition 4.2. *Let (G_a^*, \cdot) be the covering group for an n -ary group (G, f) . A representation Γ of G_a^* is a representation of G , iff $\ker \Gamma \cap G \neq \emptyset$. If Γ is irreducible G^* -representation, then it is also irreducible as a representation of G .*

Now, suppose (V, p) is a G -module. For the covering group (G_p^*, \cdot) of (G, f) we can define an action of G_p^* on V as

$$\langle x, k \rangle . v = x.v.$$

Then, it can be easily verified that V is a G_p^* -module. But, we know that $G_a^* \cong G_p^*$, so let $h : G_a^* \rightarrow G_p^*$ be any isomorphism. For any $x \in G_a^*$, define $x.v = h(x).v$. Hence V becomes a G_a^* -module. Further, if W is a G -submodule of G , then it is also a G_p^* -submodule and so a G_a^* -submodule. Hence, we proved

Theorem 4.3. *There is a bijection between the set of all irreducible representations of (G, f) and the set of all irreducible representations of G_a^* with kernels not disjoint from G .*

5. NORMAL SUBGROUPS IN POLYADIC GROUPS

In this section, we show that the representation theory of n -ary groups reduces to the representation theory of binary groups. For this we introduce the concept of normal n -ary subgroup.

Definition 5.1. An n -ary subgroup H of an n -ary group (G, f) is called *normal* if

$$f(\overset{(n-3)}{a}, \bar{a}, h, a) \in H$$

for all $h \in H$ and $a \in G$. A normal subgroup $H \neq G$ containing at least two elements is called *proper*. If G has no any proper normal subgroup, then we say that it is *simple*. If $H = G$ is the only simple subgroup of G , then we say it is *strongly simple*.

Definition 5.2. For any n -ary subgroup H of an n -ary group (G, f) we define the relation \sim_H on G , by

$$a \sim_H b \iff \exists x, y \in H : b = f(a, \overset{(n-2)}{x}, y).$$

Lemma 5.3. $a \sim_H b \iff \exists x_2, \dots, x_n \in H : b = f(a, x_2^n).$

Proof. Indeed, if $b = f(a, x_2^n)$ for some $x_2, \dots, x_n \in H$, then, in view of Theorem 1.1, for every $x \in H$ we have

$$b = f(a, x_2^n) = f(a, f(\overset{(n-2)}{x}, \bar{x}, x_2), x_3^n) = f(a, \overset{(n-2)}{x}, y),$$

where $y = f(\bar{x}, x_2^n) \in H$, so $a \sim_H b$. The converse is obvious. \square

Now it is easy to see that such defined relation is an equivalence on G . The equivalence class of G , containing a is denoted by aH and is called the *left coset* of H with the representative a . By Lemma 5.3 it has the form

$$aH = \{f(a, \overset{(n-2)}{x}, y) : x, y \in H\} = \{f(a, h_2^n) : h_2, \dots, h_n \in H\}.$$

The n -ary group (G, f) is partitioned by cosets of H .

Proposition 5.4. If H is a finite n -ary subgroup of (G, f) , then for all $a \in G$, we have $|aH| = |H|$.

Proof. By Theorem 1.2, for an n -ary group (G, f) there is a binary group (G, \cdot) , $\varphi \in \text{Aut}(G, \cdot)$ and an element $b \in G$ such that

$$f(x_1^n) = x_1 \cdot \varphi(x_2) \cdot \varphi^2(x_3) \dots \varphi^{n-1}(x_n) \cdot b,$$

for all $x_1, \dots, x_n \in G$. So, we have

$$aH = \{a \cdot \varphi(x_2) \cdot \varphi^2(x_3) \dots \varphi^{n-1}(x_n) \cdot b : x_2, \dots, x_n \in H\}.$$

But, clearly this set is in one-one correspondence with the set

$$\{\varphi(x_2) \cdot \varphi^2(x_3) \dots \varphi^{n-1}(x_n) \cdot b : x_2, \dots, x_n \in H\},$$

which does not depend on a . So, we have $|aH| = |H|$. \square

On the set $G/H = \{aH : a \in G\}$ we introduce the operation

$$f_H(a_1H, a_2H, \dots, a_nH) = f(a_1^n)H.$$

Proposition 5.5. If H is a normal n -ary subgroup of (G, f) , then $(G/H, f_H)$ is an n -ary group derived from the group $\text{Ret}_H(G/H, f)$.

Proof. First we show that the operation f_H is well-defined. For this let $a_i H = b_i H$ for some $a_i, b_i \in G$, $i = 1, 2, \dots, n$. Then

$$b_1 = f(a_1, x_2^n), \quad b_2 = f(a_2, y_2^n), \quad \dots, \quad b_n = f(a_n, z_2^n)$$

for some $x_i, y_i, \dots, z_i \in H$

Now, using Theorem 1.2 we obtain

$$\begin{aligned} f(b_1^n) &= f(f(a_1, x_2^n), f(a_2, y_2^n), \dots, f(a_n, z_2^n)) \\ &= f(f(a_1, x_2^{n-1}, f(\overset{(n-2)}{a_2}, \bar{a}_2, x_n)), f(a_2, y_2^n), \dots, f(a_n, z_1^n)) \\ &= f(f(a_1, x_2^{n-1}, a_2), f(f(\overset{(n-3)}{a_2}, \bar{a}_2, x_n, a_2), y_2^n), \dots, f(a_n, z_n^n)) \\ &= f(f(a_1, x_2^{n-1}, a_2), f(w_n, y_2^n), \dots, f(a_n, z_1^n)) \\ &= f(f(a_1, x_2^{n-2}, f(\overset{(n-2)}{a_2}, \bar{a}_2, x_{n-1}), a_2), f(w_n, y_2^n), \dots, f(a_n, z_2^n)) \\ &= f(f(a_1, x_2^{n-2}, a_2, f(\overset{(n-3)}{a_2}, \bar{a}_2, x_{n-1}, a_2)), f(w_n, y_2^n), \dots, f(a_n, z_2^n)) \\ &= f(f(a_1, x_2^{n-2}, a_2, w_{n-1}), f(w_n, y_2^n), \dots, f(a_n, z_2^n)) \\ &\vdots \\ &= f(f(a_1, a_2, w_3^{n-1}), f(w_n, y_2^n), \dots, f(a_n, z_2^n)), \end{aligned}$$

where $w_i = f(\overset{(n-3)}{a_2}, \bar{a}_2, x_i, a_2) \in H$.

Repeating this procedure for a_3, a_4 and so on, we obtain

$$f(b_1^n) = f(f(a_1^n), h_2^n).$$

This means that the operation f_H is well-defined.

It is easy to verify that $(G/H, f_H)$ is an n -ary group. Using the above procedure it is not difficult to see that H is the identity of G/H . Hence an n -ary group G/H is derived from the group $\text{Ret}_H(G/H)$. \square

Now, we return to the representations, again. Consider a representation $\Lambda : (G, f) \rightarrow GL(V)$. It is easy to see that $\ker \Lambda$ is a normal subgroup of G . Let H be a normal n -ary subgroup of (G, f) such that $H \subseteq \ker \Lambda$. Then, there is a representation $\bar{\Lambda} : G/H \rightarrow GL(V)$ such that

$$\bar{\Lambda}(aH) = \Lambda(a).$$

Conversely, from every representation of G/H , we obtain a representation of G . On the other hand, G/H is of reduced type, and hence its representations are the same as the ordinary representations of $\text{Ret}_H(G/H)$. So, we proved,

Proposition 5.6. *There is a bijection between ordinary representations of $\text{Ret}_H(G/H)$ and the set of representations of G with the property $H \subseteq \ker \Lambda$.*

Proposition 5.7. *A simple n -ary group which is not strongly simple is b -derived from an abelian group or it is reducible to a non-abelian group.*

Proof. Suppose $H = \{p\}$ is a normal n -ary subgroup of (G, f) . Then we have

$$f(p, p, \dots, p) = p, \quad \bar{p} = p, \quad \forall x \in G : f(\overset{(n-3)}{x}, \bar{x}, p, x) = p.$$

Hence

$$\begin{aligned} f(p, x_2^n) &= f(f(\overset{(n-2)}{x_2}, \bar{x}_2, p), x_2^n) \\ &= f(x_2, f(\overset{(n-3)}{x_2}, \bar{x}_2, p, x_2), x_3^n) \\ &= f(x_2, p, x_3^n). \end{aligned}$$

This shows that p is a central element and, according to [8], an n -ary group (G, f) is b -derived from a binary group (G, \cdot) . Hence, $Z(G, f) = Z(G, \cdot)$ is a normal n -ary subgroup of (G, f) . But G has no proper normal subgroups, so there are two cases:

- (1) $Z(G, \cdot) = G$ and so (G, f) is b -derived from an abelian group,
- (2) $Z(G, \cdot)$ is singleton and hence $b = 1$. In this case (G, f) is reducible to a non-abelian group (G, \cdot) .

□

Remark 5.8. To find representations of an n -ary group (G, f) , we have four cases, as follow,

- (1) only $H = G$ is a normal subgroup of (G, f) , (in this case (G, f) has only trivial representation),
- (2) (G, f) is b -derived from an abelian group,
- (3) $(G, f) = \text{der}(G, \cdot)$, (in this case representations of (G, f) are the same as the representations of (G, \cdot)),
- (4) (G, f) has proper normal n -ary subgroups, (in this case, if we know the set of normal n -ary subgroups of (G, f) , then we obtain all its representations from representations of the groups $\text{Ret}_H(G/H)$).

Finally, summarizing results of this section, we have the following theorem:

Theorem 5.9. *Representation theory of n -ary groups, reduces to the following three problems,*

- a) *representations of b -derived ternary groups from abelian groups,*
- b) *determining all normal n -ary subgroup,*
- c) *representation theory of ordinary groups.*

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